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On t-Intuitionistic Fuzzy PMS-Subalgebras of a PMS Algebra

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ABSTRACT

In this paper, the authors extend the concept of a t-intuitionistic fuzzy set to PMS-subalgebras of PMS-algebras. The authors define the t-intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and show that any intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is a t-intuitionistic fuzzy PMS-subalgebra. The authors provide the condition for an intuitionistic fuzzy set in a PMS-algebra to be a t-intuitionistic fuzzy PMS-subalgebra. The authors use their (α,β) level cuts to characterize the t-intuitionistic fuzzy PMS-subalgebras of PMS-algebra. The authors investigate whether the homomorphic images and inverse images of t-intuitionistic fuzzy PMS-subalgebras are also t-intuitionistic fuzzy PMS-subalgebras. Furthermore, the authors show that the homomorphic images and inverse images of the nonempty (α,β) level cuts of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra are again PMS-subalgebras of a PMS-algebra. Finally, the authors show that the Cartesian product of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra is itself a t-intuitionistic fuzzy PMS-subalgebra and characterize it in terms of its (α,β) level cuts.

KEYWORDS

Cartesian product, Homomorphism, Intuitionistic Fuzzy Set, level cuts, PMS-algebra, t-Intuitionistic fuzzy PMS-subalgebra, PMS-subalgebra, t-Intuitionistic Fuzzy Set

1. INTRODUCTION

In 1965, Zadeh introduced the idea of a fuzzy set as the generalization of the crisp set for describing uncertainty in our universe. Rosenfeld (1971) introduced the concept of fuzzy subgroups and established some related results. Atanassov (1986, 1989) developed the theory of an intuitionistic fuzzy set as an extension of a fuzzy set for describing uncertainties more efficiently. Since then, several authors have applied the idea of an intuitionistic fuzzy set to different algebraic structures. Biswas (1989) studied intuitionistic fuzzy subgroups of a group using the concept of intuitionistic fuzzy sets. Peng (2012) introduced the notion of intuitionistic fuzzy B-algebras in B-algebra and investigated various aspects of their homomorphic image and inverse image. Jana et al. (2015) investigated several properties of G-subalgebras of G-algebras using the concept of intuitionistic fuzzy sets.

Many researchers have also applied the idea of an intuitionistic fuzzy set for describing uncertainties in real-life situations. Yu and Li. (2022) proposed a novel intuitionistic fuzzy goal programming method for heterogeneous multi-attribute decision making under multi-source

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information. Yu et al. (2021) developed a new and unified intuitionistic fuzzy multi-objective linear programming model for portfolio selection problems to solve multi-objective decision problems with hesitation degrees and reduce the complexity of the nondeterministic polynomial-hard problems. Li and Wan (2017) developed an effective method for solving intuitionistic fuzzy multi-attribute decision-making problems with incomplete weight information. Bhaumik et al. (2017) studied a matrix game with triangular intuitionistic fuzzy numbers as payoffs and used robust ranking approaches to rank fuzzy numbers in order to solve the matrix game. Moreover, to deal with complex decision-making problems in which the membership and non-membership degrees of fuzzy concepts cannot be expressed with exact numerical values due to a lack of information in many real-life situations, Atanassov and Gargov (1989) developed an interval-valued intuitionistic fuzzy set characterized by interval-valued membership and non-membership functions rather than real numbers. Using the concept of an interval-valued intuitionistic fuzzy set, Wei et al. (2021) developed and applied an information-based score function of the interval-valued intuitionistic fuzzy set to multiattribute decision-making to overcome the limitations of existing ranking methods and rank the interval-valued intuitionistic fuzzy set well. Their results demonstrated that the information-based score function is more reasonable than existing ranking methods. Li (2011) developed the representation theorem and extension principles for interval-valued intuitionistic fuzzy sets based on the concept of level sets of interval-valued intuitionistic fuzzy sets.

Sharma (2012) developed the concept of the t-intuitionistic fuzzy set as an extension of the intuitionistic fuzzy set to deal with uncertainty and vagueness and then introduced the idea of t-intuitionistic fuzzy subgroups and t-intuitionistic fuzzy quotient groups, as well as t-intuitionistic fuzzy subring of a ring. Shuaib et al. (2020) introduced the notion of η-intuitionistic fuzzy subgroup over η-intuitionistic fuzzy subset and studied some algebraic aspects of η-intuitionistic fuzzy subgroups. Barbhuiya (2015) introduced the concepts of t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra of a BG-algebra, and then investigated the homomorphic image and inverse image of both t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra of a BG-algebra.

Iseki and Tanaka (1978) introduced a class of abstract algebra called BCK-algebra. Iseki (1980) introduced another class of abstract algebra called BCI-algebra as a generalization of BCK-algebra. Selvam and Nagalakshmi (2016) introduced a new concept, called PMS-algebra, which is related to several classes of algebra such as BCI-algebra, BCK-algebras, TM-algebra and so on. The concept of a fuzzy PMS-subalgebra of a PMS-algebra was introduced by Selvam and Nagalakshmi in 2016. The study of intuitionistic fuzzification of PMS-subalgebra and PMS-ideal of PMS-algebras was done by Derseh et al. in (2021, 2022). The authors (2022) also studied the intuitionistic Q-fuzzy PMS-subalgebra of a PMS-algebra and the intuitionistic Q-fuzzy PMS-ideals of a PMS-algebra. The notion of t-intuitionistic fuzzy subalgebra has been studied in several algebraic structures (Barbhuiya 2015; Gulzar et al. 2020**;** Sharma 2012; Shuaib et al. 2019, 2020). However, as far as we know, no study has been conducted on the t-intuitionistic fuzzification of the PMS-subalgebra of a PMS-algebra. This motivated us to develop the t-intuitionistic fuzzy PMS-subalgebras in PMS-algebras.

In this article, we apply the concept of a t-intuitionistic fuzzy set to PMS-subalgebras in PMS-algebras. We define the t-intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and show that an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is a t-intuitionistic fuzzy PMS-subalgebra. We give the condition for an intuitionistic fuzzy set in a PMS-algebra to be a t-intuitionistic fuzzy PMS-subalgebra. We characterize the t-intuitionistic fuzzy PMS-subalgebras of PMS-algebra by using their $\left(\pm, {}^{2}\right)$ level cuts. We investigate whether the homomorphic images and inverse images of t-intuitionistic fuzzy PMS-subalgebras are also t-intuitionistic fuzzy PMS-subalgebras. Moreover, we prove that the homomorphic images and inverse images of the nonempty $\left(\pm, {}^{2}\right)$ level cuts of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra are again PMS-subalgebras of a PMS-algebra. Finally, we demonstrate that the Cartesian product of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra is again t-intuitionistic fuzzy PMS-subalgebra and subsequently characterize it in terms of its (α, β) cuts.

2. PRELIMENARIES

In this section, we consider some basic definitions, results and important concepts of PMS-algebras that are needed for our work.

Definition 2.1 (**Selvam and Nagalakshmi, 2016**) A PMS-algebra is a nonempty set X with a constant 0 and a binary operation $*$ of type (2, 0) satisfying the following axioms:

i. $0 * x = x$
ii. $(y * x) * (z * x) = z * y$, for all $x, y, z \in X$

We can define a binary relation $\leq$ *in* X *by* $x \leq y$ *if and only if* $x * y = 0$.

Definition 2.2 (**Selvam and Nagalakshmi, 2016**) *A nonempty subset S of a PMS-algebra is called a PMS-sub algebra of X if* $x * y \in S$, *for all* $x, y \in S$.

Proposition 2.3 (**Selvam and Nagalakshmi, 2016**) *In any PMS-algebra* $(X, *, 0)$ *the following properties hold for all* $x, y, z \in X$,

i. $x * x = 0$
ii. $(y * x) * x = y$
iii. $x * (y * x) = y * 0$
iv. $(y * x) * z = (z * x) * y$
v. $(x * y) * 0 = y * x = (0 * y) * (0 * x)$

Definition 2.4 (**Zadeh, 1965**) *A fuzzy subset* A *in a nonempty set* X *is characterized by the mapping* $\mu_A : X \to [0,1]$ *where* $\mu_A(x)$ *defines the degree of membership of the element* x *in* X.

Definition 2.5 (**Atanassov, 1986, 1989**) *An intuitionistic fuzzy set A in a nonempty set* X *is an object having the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, *where the functions* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *define the degree of membership and the degree of nonmembership respectively, with the condition* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for all* $x \in X$.

Definition 2.6 (**Atanassov, 1986, 1989**) *Let A and B be two intuitionistic fuzzy subsets of the set* X, *where* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$, *then*

$$A \cap B = \left\{x, min\left\{\mu_A(x), \mu_B(x)\right\}, max\left\{\nu_A(x), \nu_B(x)\right\} \mid x \in X\right\}$$
$$A \cup B = \left\{x, max\left\{\mu_A(x), \mu_B(x)\right\}, min\left\{\nu_A(x), \nu_B(x)\right\} \mid x \in X\right\}.$$
$$\bar{A} = \{x, \nu_A(x), \mu_A(x) \mid x \in X\}$$
$$\Box A = \left\{x, \mu_A(x), 1 - \mu_A(x) \mid x \in X\right\}$$
$$\Diamond A = \left\{x, 1 - \nu_A(x), \nu_A(x) \mid x \in X\right\}.$$

Definition 2.7 (**Derseh et al., 2021**) *An intutionistic fuzzy subset* $A = (\mu_A, \nu_A)$ *in a PMS-algebra* X *is called an intuitionistic fuzzy PMS-subalgebra of* X *if*

$$\mu_A(x * y) \geq min\left\{\mu_A(x), \mu_A(y)\right\} \text{ and } \nu_A(x * y) \leq max\left\{\nu_A(x), \nu_A(y)\right\}, \forall\ x, y \in X$$

Definition 2.8 (**Mostafa et al., 2015**) *Let A be a fuzzy set in a nonempty set* X *with a membership function* $\mu_A : X \to [0,1]$ *and let* $t \in [0,1]$. *Then the fuzzy set* A^t *in* X *is called the* t*-fuzzy subset* *of* X *whose membership function is* μ_{A^t} *(w.r.t fuzzy set* A *) and is defined by* $\mu_{A^t}(x) = min\{\mu_A(x), t\}$ *, for all* $x \in X$.

Definition 2.9 (Sharma, 2012) *Let* $A = (\mu_A, \nu_A)$ *be an intuitionistic fuzzy set in a nonempty set* X *and* $t \in [0,1]$. *Then the t-intuitionistic fuzzy set (t-IFS)* A^t *in a nonempty set* X *is an object having the form* $A^t = \{\langle x, \mu_{A^t}(x), \nu_{A^t}(x)\rangle \mid x \in X\}$, *where the function* $\mu_{A^t} : X \to [0,1]$ *and* $\nu_{A^t} : X \to [0,1]$ *denote the degree of membership and degree of nonmembership, respectively such that* $\mu_{A^t}(x) = min\{\mu_A(x), t\}$ *and* $\nu_{A^t}(x) = max\{\nu_A, 1-t\}$ *satisfying the condition* $0 \le \mu_{A^t}(x) + \nu_{A^t}(x) \le 1$, *for all* $x \in X$.

Note: For the sake of Simplicity we shall use the symbol $A^t = (\mu_{A^t}, \nu_{A^t})$, for t-IFS $A^t = \{\langle x, \mu_{A^t}(x), \nu_{A^t}(x)\rangle \mid x \in X\}$.

Remark 2.1 *Let* $A^t = \{\langle x, \mu_{A^t}(x), \nu_{A^t}(x)\rangle \mid x \in X\}$ *be a* t *-IFSs of the set* X . *Then*

$\Box A^t = \{x, \mu_{A^t}(x), 1 - \mu_{A^t}(x) \mid x \in X\} = \{x, \mu_{A^t}(x), \bar{\mu}_{A^t}(x) \mid x \in X\}$ and

$\Diamond A^t = \{x, 1 - \nu_{A^t}(x), \nu_{A^t}(x) \mid x \in X\} = \{x, \bar{\nu}_{A^t}(x), \nu_{A^t}(x) \mid x \in X\}$

Remark 2.2 (Sharma, 2012, Shuaib et al., 2019) *Let* $A^t = (\mu_{A^t}, \nu_{A^t})$ *and* $B^t = (\mu_{B^t}, \nu_{B^t})$ *be any two* t *-intuitionistic fuzzy subsets of any nonempty set* X , *then* $(A \cap B)^t = A^t \cap B^t$ *and* $(A \cup B)^t = A^t \cup B^t$.

Definition 2.10 (Sharma, 2011) *Let* X *and* Y *be PMS-algebras. The mapping* $f : X \to Y$ *is called a homomorphism of PMS-algebras if* $f(x * y) = f(x) * f(y)$, *for all* $x, y \in X$

Remark 2.3 (Sharma, 2012) *Let* $f : X \to Y$ *be a mapping and A such that B are any two* t *-IFSs of X and Y respectively, then*

$f(A^t) = (f(A))^t$ *and* $f^{-1}(B^t) = (f^{-1}(B))^t$, *for all* $t \in [0,1]$.

Definition 2.11 (Gulzar et al., 2020) *Let* $A^t = (\mu_{A^t}, \mu_{A^t})$ *and* $A^t = (\mu_{A^t}, \mu_{A^t})$ *be two* t *-intuitionistic fuzzy subsets of* X *and* Y *respectively. Then their cartesian product of* A^t *and* B^t *denoted by* $A^t \times B^t$ *is defined as* $A^t \times B^t = \{\langle (x, y), \mu_{A^t \times B^t}, \nu_{A^t \times B^t}\rangle \mid x \in X$ *and* $y \in Y\}$, *where* $\mu_{A^t \times B^t}(x * y) = min\{\mu_{A^t}(x)\}, \mu_{A^t}(y))$ *and* $\nu_{A^t \times B^t}(x * y) = max\{\nu_{A^t}(x)\}, \nu_{A^t}(y)$, *for all* $x \in X$ *and* $y \in Y$.

Remark 2.4 *Let* X *and* Y *be any two PMS-algebras, for every* $(x, y), (u, v) \in X \times Y$, *we define* '$*$' *on* $X \times Y$ *by* $(x, y) * (u, v) = (x * u, y * v)$. *Clearly* $(X \times Y; *, (0,0))$ *is a PMS-algebra.*

Definition 2.12 (Gulzar et al., 2020) *Let* A^t *be* t *-IFS of X w.r.t IFS A. Then the* (α, β) *-cut of* A^t *is a crisp subset* $C_{(\alpha,\beta)}(A^t)$ *of* X *and is given by* $C_{(\alpha,\beta)}(A^t) = \{x \in X : \mu_{A^t}(x) \ge \alpha, \nu_{A^t}(x) \le \beta\}$, *where* $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \le 1$.

3. t-INTUITIONISTIC FUZZY PMS-SUBALGEBRS OF A PMS-ALGEBRA

In this section, we study the notion of t-intuitionistic fuzzy PMS-subalgebra defined on intuitionistic fuzzy subset of X and show that every intuitionistic fuzzy PMS-subalgebra is also t-intuitionistic fuzzy PMS-subalgebra, but the converse need not be true.We also establish many fundamental results. In this and the next sections X and Y denote PMS-algebra unless otherwise specified.

Definition 3.1 *Let $t \in [0,1]$. A t-IFS A^t of X is called a t-intuitionistic fuzzy PMS-subalgebra (t-IF PMS-SA) of a PMS-algebra X if $\mu_{A^t}(x * y) \geq min\{\mu_{A^t}(x), \mu_{A^t}(y)\}$ and $\nu_{A^t}(x * y) \leq max\{\nu_{A^t}(x), \nu_{A^t}(y)\}$, for all $x, y \in X$.*

Lemma 3.2 *If A is an IF PMS-SA of a PMS-algebra X, then $\mu_{A^t}(0) \geq \mu_{A^t}(x)$ and $\nu_{A^t}(0) \leq \nu_{A^t}(x)$ for all $x \in X$.*

Proof. Since A is an IF PMS-SA of a PMS-algebra X, then

$$\begin{aligned}
&\mu_{A^t}(0) = \mu_{A^t}(x*x) = \min\{\mu_A(x*x), t\} \\
&\geq \min\{\min\{\mu_A(x), \mu_A(x)\}, t\} \\
&= min\{\mu_A(x), t\} = \mu_{A^t}(x)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_{A^t}(0) = \nu_{A^t}(x*x) = \max\{\nu_A(x*x), 1-t\} \\
&\leq \max\{\max\{\nu_A(x), \nu_A(x)\}, t\} \\
&= max\{\nu_A(x), 1-t\} = \nu_{A^t}(x).
\end{aligned}$$

Lemma 3.3 *Let A^t be a t-IF PMS-SA of a PMS-algebra X. If $x \leq y$, then*

$$\mu_{A^t}(x) = \mu_{A^t}(y) \text{ and } \nu_{A^t}(x) = \nu_{A^t}(y),\ \forall\ x, y \in X.$$

Proof. Let $x, y \in X.$ such that $x \leq y$. Then $x*y = 0$. By Definition 2.1 (i) and Proposition 2.3 (iv), we have $\mu_{A^t}(y) = \mu_{A^t}(0*y) = \mu_{A^t}((x*y)*y) = \mu_{A^t}(x)$ and $\nu_{A^t}(y) = \nu_{A^t}(0*y) = \mu_{A^t}((x*y)*y) = \nu_{A^t}(x)$

Theorem 3.4 *If A is an IF PMS-SA of X, then A^t is also a t-IF PMS-SA of X.*

Proof. Let A be an IF PMS-SA of X and $x, y \in X$. Then by the definition of t-IFS,

$$\begin{aligned}
&\mu_{A^t}(x*y) = \min\{\mu_A(x*y), t\} \\
&\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, t\} \\
&= min\{min\{\mu_A(x), t\}, min\{\mu_A(y), t\}\}
\end{aligned}$$

$$= min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\}$$
$$\Rightarrow \mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\}$$

And

$$\nu_{A^t}\left(x*y\right) = max\left\{\nu_A\left(x*y\right),1-t\right\}$$
$$\leq max\left\{max\left\{\nu_A\left(x\right),\nu_A\left(y\right)\right\},1-t\right\}$$
$$= max\left\{max\left\{\mu_A\left(x\right),1-t\right\},max\left\{\nu_A\left(y\right),1-t\right\}\right\}$$
$$= \max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$$
$$\Rightarrow \nu_{A^t}\left(x*y\right) \leq \max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$$

Therfore $\mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\} and\, \nu_{A^t}\left(x*y\right) \leq max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\} for\, all\, x,y \in X.$

Remark 3.1 The converse of above Theorem need not be neccassarily true. This fact is shown by the following Example:

Example 3.5 Consider $X = \left\{0,1,2\right\}$ such that $\left(X,*,0\right)$ is a PMS-algebra with the following table

Table 1

*	**0**	**1**	**2**
0	0	1	2
1	2	0	1
2	1	2	0

Define the intuitionistic fuzzy subset $\left(\mu_A,\nu_A\right)$ in X by

$$\mu_A\left(x\right) = \begin{cases} 0.5 & if\ x = 0, \\ 0.6 & if\ x = 1, \\ 0.4 & if\ x = 2, \end{cases} \text{ and } \nu_A\left(x,q\right) = \begin{cases} 0.4 & if\ x = 0 \\ 0.3 & if\ x = 1, \\ 0.6 & if\ x = 2, \end{cases} \text{for all } x \in \text{X}.$$

Since $\mu_A\left(0\right) = 0.5 < 0.6 = \mu_A\left(1\right)$ and $\nu_A\left(0\right) = 0.4 > 0.3 = \nu_A\left(1\right)$. It follows that A is not an intuitionistic fuzzy PMS-subalgebra of X. Take $t = 0.3$. Then, $\mu_A\left(x\right) > t,\ \forall x \in X$.

Also, $\nu_A\left(x\right) < 1-t$, for all $x \in X$. Therefore $\mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\}$ and $\nu_{A^t}\left(x*y\right) \leq max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$, for all $x \in X$.

Hence μ_{A^t} is a t-IF PMS-SA of X.

Theorem 3.6 *Let* A^t *be a t-IF PMS-SA of* X *and* $x \in X$. *Then* $\mu_{A^t}(x * y) = \mu_{A^t}(y)$ *and* $\nu_{A^t}(x * y) = \nu_{A^t}(y)$ *for all* $y \in X$ *if and only if* $\mu_{A^t}(x) = \mu_{A^t}(0)$ *and* $\nu_{A^t}(x) = \nu_{A^t}(0)$.

Proof. Assume that $\mu_{A^t}(x*y) = \mu_{A^t}(y)$ and $\nu_{A^t}(x*y) = \nu_{A^t}(y)$. Then by Lemma 3.2 we have $\mu_{A^t}(0) \geq \mu_{A^t}(x)$ and $\nu_{A^t}(0) \leq \nu_{A^t}(x)$, for all $x \in X$. (1)

Also by Proposition 2.3, $\mu_{A^t}(x) = \mu_{A^t}((x*0)*0) \geq min\{\mu_{A^t}(x*0), \mu_{A^t}(0)\} = \mu_{A^t}(0)$ and $\nu_{A^t}(x) = \nu_{A^t}((x*0)*0) \leq max\{\nu_{A^t}(x*0), \nu_{A^t}(0)\} = \nu_{A^t}(0)$.

Hence $\mu_{A^t}(x) \geq \mu_{A^t}(0)$ and $\nu_{A^t}(x) \leq \nu_{A^t}(0)$ (2)

Therefore from (1) and (2),we have $\mu_{A^t}(x) = \mu_{A^t}(0)$ and $\nu_{A^t}(x) = \nu_{A^t}(0)$.

Conversely, assume that $\mu_{A^t}(x) = \mu_{A^t}(0)$ and $\nu_{A^t}(x) = \nu_{A^t}(0)$. Then by Lemma 3.2 we have $\mu_{A^t}(x) \geq \mu_{A^t}(y)$ and $\nu_{A^t}(x) \leq \nu_{A^t}(y)$, $\forall y \in Y$. Since A^t is a t-IF PMS-SA of X, then $\mu_{A^t}(x*y) \geq min\{\mu_{A^t}(x), \mu_{A^t}(y)\} = \mu_{A^t}(y)$ and

$$\nu_{A^t}(x*y) \leq max\{\nu_{A^t}(x), \nu_{A^t}(y)\} = \nu_{A^t}(y)$$

Thus $\mu_{A^t}(x*y) \geq \mu_{A^t}(y)$ and $\nu_{A^t}(x*y) \leq \nu_{A^t}(y)$, for all $y \in X$. (3)

On the other hand by Proposition 2.3 (ii) and 2.4 (v) it follows that

$$\mu_{A^t}(y) = \mu_{A^t}((y*x)*x) \geq min\{\mu_{A^t}(y*x), \mu_{A^t}(x)\}$$
$$= min\{\mu_{A^t}((x*y)*0), \mu_{A^t}(x)\}$$
$$\geq min\{min\{\mu_{A^t}(x*y), \mu_{A^t}(0)\}, \mu_{A^t}(x)\}$$
$$= min\{\mu_{A^t}(x*y), \mu_{A^t}(x)\} = \mu_{A^t}(x*y)\}$$

and

$$\nu_{A^t}(y) = \nu_{A^t}((y*x)*x) \leq max\{\mu_{A^t}(y*x), \mu_{A^t}(x)\}$$
$$= max\{\nu_{A^t}((x*y)*0), \nu_{A^t}(x)\}$$
$$\leq max\{max\{\nu_{A^t}(x*y), \nu_{A^t}(0)\}, \nu_{A^t}(x)\}$$
$$= max\{\nu_{A^t}(x*y), \nu_{A^t}(x)\} = \nu_{A^t}(x*y)\}$$

Hence $\mu_{A^t}(y) \geq \mu_{A^t}(x*y) and\ \nu_{A^t}(y) \leq \nu_{A^t}(x*y)$ (4)

Therefore from (3) and (4) we have $\mu_{A^t}(x*y) = \mu_{A^t}(y)$ and $\nu_{A^t}(x*y) = \nu_{A^t}(y)$

Lemma 3.7 *If* A^t *is a t-IF PMS-SA of* X *, then* $\mu_{A^t}(x*y)=\mu_{A^t}(y*x)$ *and* $\nu_{A^t}(x*y)=\nu_{A^t}(y*x)$ *for all* $x,y\in X$

Proof. The proof of this Lemma is trivial.

Theorem 3.8 *Let* A^t *be a t-IF PMS-SA of* X *. If* $\mu_{A^t}(x*y)=\mu_{A^t}(0)$ *and* $\nu_{A^t}(x*y)=\nu_{A^t}(0)$ *for all* $x,y\in X$ *, then* $\mu_{A^t}(x)=\mu_{A^t}(y)$ *and* $\nu_{A^t}(x)=\nu_{A^t}(y)$

Proof. Let $x,y\in X$ such that $\mu_{A^t}(x*y)=\mu_{A^t}(0)$ and $\nu_{A^t}(x*y)=\nu_{A^t}(0)$. Then by Proposition 2.3 (ii), Definition 3.1 and Lemma 3.2, we have

$$\begin{aligned}\mu_{A^t}(x)&=\mu_{A^t}((x*y)*y)\\&\geq min\{\mu_{A^t}(x*y),\mu_{A^t}(y)\}\\&=min\{\mu_{A^t}(0),\mu_{A^t}(y)\}=\mu_{A^t}(y)\end{aligned}$$

$$\Rightarrow \mu_{A^t}(x)\geq\mu_{A^t}(y) \qquad (1)$$

Conversely, $\mu_{A^t}(y)=\mu_{A^t}((y*x)*x)$

$$\begin{aligned}&\geq min\{\mu_{A^t}(y*x),\mu_{A^t}(x)\}\\&=min\{\mu_{A^t}(x*y),\mu_{A^t}(x)\}\ (\text{By Lemma3.7})\\&=min\{\mu_{A^t}(0),\mu_{A^t}(x)\}=\mu_{A^t}(x)\end{aligned}$$

$$\Rightarrow \mu_{A^t}(y)\geq\mu_{A^t}(x) \qquad (2)$$

Therefore from (1) and (2), we have $\mu_{A^t}(x)=\mu_{A^t}(y)$.

Similarly we have $\nu_{A^t}(x)=\nu_{A^t}(y)$.

Corollary 3.9 *Let* A^t *be a t-IF PMS-SA of* X *. If* $\mu_{A^t}(x*y)=1$ *and* $\nu_{A^t}(x*y)=0$ *for all* $x,y\in X$ *, then* $\mu_{A^t}(x)=\mu_{A^t}(y)$ *and* $\nu_{A^t}(x)=\nu_{A^t}(y)$

Proof. It can be done similar to Theorem 3.8.

The result of the next theorem establishes the condition under which an intuitionistic fuzzy set in a PMS-algebra is a t-intuitionistic fuzzy PMS-subalgebra.

Theorem 3.10 *Let* A *be an IFS of a PMS-algebra* X *and* $t\in[0,1]$ *such that* $t\leq min\{m,1-n\}$, *where* $m=min\{\mu_A(x)\mid x\in X\}$ *and* $n=max\{\nu_A(x)\mid x\in X\}$ *. Then* A^t *is a* t*-IF PMS-SA of* X *.*

Proof. Since $t\leq min\{m,1-n\}$, we have $m\geq t$ and $1-n\geq t$

$\Rightarrow m\geq t$ and $n\leq 1-t$

$\Rightarrow min\{\mu_A(x)\mid x\in X\}\geq t$ and $max\{\nu_A(x)\mid x\in X\}\leq 1-t$

Since $\mu_A\left(x\right) \geq min\{\mu_A\left(x\right) \mid x \in X\}$ and $\nu_A\left(x\right) \leq max\{\nu_A\left(x\right) \mid x \in X\}$, it follows that

$\mu_A\left(x\right) \geq t$ and $\nu_A\left(x\right) \leq 1-t$, for all $x \in X$.

$\Rightarrow min\left\{\mu_A\left(x\right), t\right\} = t$ and $max\left\{\nu_A\left(x\right), 1-t\right\} = 1-t$

$\Rightarrow \mu_{A^t}\left(x\right) = t$ and $\nu_{A^t}\left(x\right) = 1-t$, for all $x \in X$.

Thus, $\mu_{A^t}\left(x*y\right) = t \geq min\left\{t,t\right\} = min\left\{\mu_{A^t}\left(x\right), \mu_{A^t}\left(y\right)\right\}$ and

$\nu_{A^t}\left(x*y\right) = 1-t \leq max\left\{1-t, 1-t\right\} = max\left\{\nu_{A^t}\left(x\right), \nu_{A^t}\left(y\right)\right\}$, for all $x,y \in X$.

$\Rightarrow \mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right), \mu_{A^t}\left(y\right)\right\}$ and $\nu_{A^t}\left(x*y\right) \leq max\left\{\nu_{A^t}\left(x\right), \nu_{A^t}\left(y\right)\right\}, \forall x,y \in X.$

Therefore A^t is a t-IF PMS-SA of X.

Theorem 3.11 The intersection of any two t-IF PMS-SAs of X is also a t-IF PMS-SA of X.

Proof. Let A^t and B^t be any two t-IF PMS-SAs of a PMS-algebra X and $x, y \in X$. Then

$$\mu_{(A\cap B)^t}\left(x*y\right) = min\left\{\mu_{A\cap B}\left(x*y\right), t\right\}$$

$$= min\left\{\min\left\{\mu_A\left(x*y\right), \mu_B\left(x*y\right)\right\}, t\right\}$$

$$= min\left\{\min\left\{\mu_A\left(x*y\right), t\right\}, min\left\{\mu_B\left(x*y\right), t\right\}\right\}$$

$$= min\left\{\mu_{A^t}\left(x*y\right), \mu_{B^t}\left(x*y\right)\right\}$$

$$\geq min\left\{min\left\{\mu_{A^t}\left(x\right), \mu_{A^t}\left(y\right)\right\}, min\left\{\mu_{B^t}\left(x\right), \mu_{B^t}\left(y\right)\right\}\right\}$$

$$= min\left\{min\left\{\mu_{A^t}\left(x\right), \mu_{B^t}\left(x\right)\right\}, min\left\{\mu_{A^t}\left(y\right), \mu_{B^t}\left(y\right)\right\}\right\}$$

$$= min\left\{\mu_{A^t\cap B^t}\left(x\right), \mu_{A^t\cap B^t}\left(y\right)\right\}$$

$$= min\left\{\mu_{(A\cap B)^t}\left(x\right), \mu_{(A\cap B)^t}\left(y\right)\right\} \left(\text{By Remark 2.1}\right)$$

and

$$\nu_{(A\cap B)^t}\left(x*y\right) = max\left\{\nu_{A\cap B}\left(x*y\right), 1-t\right\}$$

$$= max\left\{\max\left\{\nu_A\left(x*y\right), \nu_B\left(x*y\right)\right\}, 1-t\right\}$$

$$= \max\left\{\max\left\{\nu_A\left(x*y\right), 1-t\right\}, \max\left\{\nu_B\left(x*y\right), 1-t\right\}\right\}$$

$$= \max\left\{\nu_{A^t}\left(x*y\right), \nu_{B^t}\left(x*y\right)\right\}$$

$$\leq \max\left\{\max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\},\max\left\{\nu_{B^t}\left(x\right),\nu_{B^t}\left(y\right)\right\}\right\}$$

$$= max\left\{max\left\{\nu_{A^t}\left(x\right),\nu_{B^t}\left(x\right)\right\},max\left\{\nu_{A^t}\left(y\right),\nu_{B^t}\left(y\right)\right\}\right\}$$

$$= \max\left\{\nu_{A^t\cap B^t}\left(x\right),\nu_{A^t\cap B^t}\left(y\right)\right\}$$

$$= max\left\{\nu_{(A\cap B)^t}\left(x\right),\nu_{(A\cap B)^t}\left(y\right)\right\} \left(\text{By Remark 2.1}\right)$$

$$\Rightarrow \mu_{A^t\cap B^t}\left(x*y\right) \geq min\left\{\mu_{A^t\cap B^t}\left(x\right),\mu_{A^t\cap B^t}\left(y\right)\right\}\text{and}\nu_{A^t\cap B^t}\left(x*y\right) \leq max\left\{\nu_{A^t\cap B^t}\left(x\right),\nu_{A^t\cap B^t}\left(y\right)\right\}.$$

Therefore $A^t \cap B^t$ is a t-IF PMS-SA of X.

The above theorem can also be generalized to any family of t-IF PMS-SAs of a PMS-algebra X as given in the next corollary.

Corollary 3.12 *Intersection of a family of t-IF PMS-SAs of a PMS-algebra X is again a t-IF PMS-SA of X.*

Remark 3.2 *The union of any two t-IF PMS-SAs of a PMS-algebra X may not be a t-IF PMS-SA of X.This is shown by the next Example.*

Example 3.13 *Let Z be the set of all integers. Let $*$ be a binary operation on Z defined by $x * y = y - x$ for all $x, y \in Z$, where '$-$' is the usual subtraction of integers.Then $\left(Z,*,0\right)$ is a PMS-algebra.*

Let $A = \left\{x, \mu_A\left(x\right), \nu_A\left(x\right) \mid x \in Z\right\}$ and $B = \left\{x, \mu_B\left(x\right), \nu_B\left(x\right) \mid x \in Z\right\}$ be intuitionistic fuzzy sets respectively defined by

$$\mu_A\left(x\right) = \begin{cases} 0.7 & \text{if} \quad x \in 2 \\ 0.3 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \nu_A\left(x\right) = \begin{cases} 0.2 & \text{if} \quad x \in 2 \\ 0.5 & \text{otherwise} \end{cases}$$

and

$$\mu_B\left(x\right) = \begin{cases} 0.5 & \text{if} \quad x \in 3 \\ 0.2 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \nu_B\left(x\right) = \begin{cases} 0.4 & \text{if} \quad x \in 3 \\ 0.5 & \text{otherwise} \end{cases}$$

Clearly, A and B are IF PMS-SAs of Z. Thus by Theorem 3.4 A^t and B^t are t-IF PMS-SAs of Z for $t \in \left[0,1\right]$.

Now $\mu_{A\cup B}\left(x\right) = max\left\{\mu_A\left(x\right),\mu_B\left(x\right)\right\}$ and $\nu_{A\cup B}\left(x\right) = min\left\{\nu_A\left(x\right),\nu_B\left(x\right)\right\}$. Therefore

$$\mu_{A\cup B}\left(x\right) = \begin{cases} 0.7 & \text{if} \quad x \in 2 \\ 0.5 & \text{if} \quad x \in 3-2 \\ 0.3 & \text{if} \quad x \notin 2\text{and } x \notin 3 \end{cases} \qquad \text{and}\nu_{A\cup B}\left(x\right) = \begin{cases} 0.2 & \text{if} \quad x \in 2 \\ 0.4 & \text{if} \quad x \in 3-2 \\ 0.5 & \text{if} \quad x \notin 2\text{and } x \notin 3 \end{cases}$$

If we take $x = 2$ and $y = 3$, then $\mu_{A\cup B}\left(x\right) = 0.7, \mu_{A\cup B}\left(y\right) = 0.5, \nu_{A\cup B}\left(x\right) = 0.2,$ and $\nu_{A\cup B}\left(y\right) = 0.4$. Now $\mu_{A\cup B}\left(x*y\right) = \mu_{A\cup B}\left(y-x\right) = \mu_{A\cup B}\left(1\right) = 0.3$ and

$$min\left\{\mu_{A\cup B}\left(x\right),\mu_{A\cup B}\left(y\right)\right\} = min\left\{0.7,0.5\right\} = 0.5 .$$

$$\Rightarrow \mu_{A\cup B}\left(x^*y\right) = 0.3 \ngeq 0.5 = min\left\{\mu_{A\cup B}\left(x\right), \mu_{A\cup B}\left(y\right)\right\} \tag{1}$$

Also, $\nu_{A\cup B}\left(x^*y\right) = \nu_{A\cup B}\left(y-x\right) = \nu_{A\cup B}\left(1\right) = 0.5$ and $max\left\{\nu_{A\cup B}\left(x\right), \nu_{A\cup B}\left(y\right)\right\} = 0.4$.

$$\Rightarrow \nu_{A\cup B}\left(x^*y\right) = 0.5 \nleq 0.4 = max\left\{\nu_{A\cup B}\left(x\right), \nu_{A\cup B}\left(y\right)\right\} \tag{2}$$

From (1) and (2) we get $\mu_{A\cup B}\left(x^*y\right) < min\left\{\mu_{A\cup B}\left(x\right), \mu_{A\cup B}\left(y\right)\right\}$ and

$\nu_{A\cup B}\left(x^*y\right) > max\left\{\nu_{A\cup B}\left(x\right), \nu_{A\cup B}\left(y\right)\right\}$ This contradicts definition 2.7.

Hence $A \cup B$ is not an IF-PMS-SA of Z.

Now take t=0.6. Then $min\left\{\mu_{A\cup B}\left(x^*y\right), 0.6\right\} = \min\left\{0.3, 0.6\right\} = 0.3$ and

$min\left\{min\left\{\mu_{A\cup B}\left(x\right), 0.6\right\}, min\left\{\mu_{A\cup B}\left(y\right), 0.6\right\}\right\} = min\left\{min\left\{0.7, 0.6\right\}, min\left\{0.5, 0.6\right\}\right\}$

$= \min\left\{0.6, 0.5\right\} = 0.5$.

Then $min\left\{\mu_{A\cup B}\left(x^*y\right), 0.6\right\} = 0.3 < 0.5 = min\left\{min\left\{\mu_{A\cup B}\left(x\right), 0.6\right\}, min\left\{\mu_{A\cup B}\left(y\right), 0.6\right\}\right\}$.

$\Rightarrow \mu_{(A\cup B)^t}\left(x^*y\right) < min\left\{\mu_{(A\cup B)^t}\left(x\right), \mu_{(A\cup B)^t}\left(y\right)\right\}$, which contradicts definition 3.1.

Hence $(A \cup B)^t = A^t \cup B^t$ is not a t-IF PMS-SA of Z.

Theorem 3.14 *If A^t is a t-IF PMS-SA of X, then $\Box A^t$ is also a t-IF PMS-SA of X.*

Proof. Suppose A^t is a t-IF PMS-SA of a PMS-algebra X.Then from Definition 3.1, we have $\mu_{A^t}\left(x^*y\right) \geq min\left\{\mu_{A^t}\left(x\right), \mu_{A^t}\left(y\right)\right\}$ for all $x, y \in X$. So it remains to show that $\bar{\mu}_{A^t}\left(x^*y\right) \leq max\left\{\bar{\mu}_{A^t}\left(x\right), \bar{\mu}_{A^t}\left(y\right)\right\}$ for all $x, y \in X$.

Now, $\bar{\mu}_{A^t}\left(x^*y\right) = 1 - \mu_{A^t}\left(x^*y\right) \leq 1 - \min\left\{\mu_{A^t}\left(x\right), \mu_{A^t}\left(y\right)\right\}$

$= max\left\{1 - \mu_{A^t}\left(x\right), 1 - \mu_{A^t}\left(y\right)\right\}$

$= max\left\{\bar{\mu}_{A^t}\left(x\right), \bar{\mu}_{A^t}\left(y\right)\right\}$

$\Rightarrow \bar{\mu}_{A^t}\left(x^*y\right) \leq max\{\bar{\mu}_{A^t}\left(x\right), \bar{\mu}_{A^t}\left(y\right)$

Hence $\Box A^t$ is a t-IF PMS-SA of X.

Theorem 3.15 *If A^t is a t-IF PMS-SA of X, then $\Diamond A^t$ is also a t-IF PMS-SA of X.*

Proof. Suppose A^t is a t-IF PMS-SA of a PMS-algebra X. Then from Definition 3.1, we have $\nu_{A^t}\left(x^*y\right) \leq max\left\{\nu_{A^t}\left(x\right), \nu_{A^t}\left(y\right)\right\}$, for all $x, y \in X$. So it suffices to show that $\bar{\nu}_{A^t}\left(x^*y\right) \geq min\left\{\bar{\nu}_{A^t}\left(x\right), \bar{\nu}_{A^t}\left(y\right)\right\}$ for all $x, y \in X$.

Now, $\overline{\nu}_{A^t}\left(x*y\right) = 1-\nu_{A^t}\left(x*y\right) \geq 1-max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$

$= min\left\{1-\nu_{A^t}\left(x\right),1-\nu_{A^t}\left(y\right)\right\}$

$= min\left\{\overline{\nu}_{A^t}\left(x\right),\overline{\nu}_{A^t}\left(y\right)\right\}$

$\Rightarrow \overline{\nu}_{A^t}\left(x*y\right) \geq min\left\{\overline{\nu}_{A^t}\left(x\right),\overline{\nu}_{A^t}\left(y\right)\right\}$

Hence $\Diamond A^t$ is a t-IF PMS-SA of X.

Theorem 3.16 *Let A^t be a t-IFS of X. Then A^t is a t-IF PMS-SA of X if and only if the nonempty subset $C_{\alpha,\beta}\left(A^t\right)$ of X is a PMS-SA of X for all $\alpha,\beta \in \left[0,1\right]$ with $\alpha+\beta \leq 1$.*

Proof. Suppose A^t is a t-IF PMS-SA of X. Let $x,y \in C_{\alpha,\beta}\left(A^t\right)$. Then

$\mu_{A^t}\left(x\right) \geq \alpha, \mu_{A^t}\left(y\right) \geq \alpha$ and $\nu_{A^t}\left(x\right) \leq \beta, \nu_{A^t}\left(y\right) \leq \beta$

$\Rightarrow \mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\} \geq \alpha$ and $\nu_{A^t}\left(x*y\right) \leq max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\} \leq \beta$

Thus $x*y \in C_{\alpha,\beta}\left(A^t\right)$. There fore, $C_{\alpha,\beta}\left(A^t\right)$ is a PMS-subalgebra of X.

Conversely, suppose $C_{\alpha,\beta}\left(A^t\right)$ is a PMS-subalgebra of X. Let $x,y \in X$ and

$\alpha = min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\}$ and $\beta = max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$.

Then $\mu_{A^t}\left(x\right) \geq \alpha,\ \nu_{A^t}\left(x\right) \leq \beta$ and $\mu_{A^t}\left(y\right) \geq \alpha,\ \nu_{A^t}\left(y\right) \leq \beta$. Thus, $x,y \in C_{\alpha,\beta}\left(A^t\right)$.

Since $C_{\alpha,\beta}\left(A^t\right)$ is a PMS-subalgebra of X, it follws that $x*y \in C_{\alpha,\beta}\left(A^t\right)$.

So, $\mu_{A^t}\left(x*y\right) \geq \alpha = min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\}$ and $\nu_{A^t}\left(x*y\right) \leq \beta = max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$

Hence $\mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\}$ and $\nu_{A^t}\left(x*y\right) \leq max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\}$

Therfore A^t is a t-IF PMS-SA of X.

Corollary 3.17 *If A^t is a t-IF PMS-SA of X, then $C_{1,0}\left(A^t\right) = \{x \in X : \mu_{A^t}\left(x\right) = 1$ and $\nu_{A^t}\left(x\right) = 0\}$ is a PMS-subalgebra of X.*

Proof. Let $x,y \in C_{1,0}\left(A^t\right)$. then $\mu_{A^t}\left(x\right) = 1 = \mu_{A^t}\left(y\right)$ and $\nu_{A^t}\left(x\right) = 0 = \nu_{A^t}\left(y\right)$

$\Rightarrow \mu_{A^t}\left(x*y\right) \geq min\left\{\mu_{A^t}\left(x\right),\mu_{A^t}\left(y\right)\right\} = 1$ and $\nu_{A^t}\left(x*y\right) \leq max\left\{\nu_{A^t}\left(x\right),\nu_{A^t}\left(y\right)\right\} = 0$

Thus, $\mu_{A^t}\left(x*y\right) = 1$ and $\nu_{A^t}\left(x*y\right) = 0$. So, $x*y \in C_{1,0}\left(A^t\right)$.

Hence $C_{1,0}\left(A^t\right)$ is a PMS-SA of X.

4. HOMOMORPHISM OF t-INTUITIONISTIC FUZZY PMS-SUBALGEBRAS

In this section, we study a t-IF PMS-SA of a PMS-algebra under homomorphism and show that the homomorphic image and inverse image of t-IF PMS-SA form a t-IF PMS-SA of a PMS-algebra.

Definition 4.1 *Let X and Y be two nonempty sets and $f: X \to Y$ be a mapping. Let A^t and B^t be t-IFS's of X and Y respectively. Then the image of A^t under f is denoted by $f\left(A^t\right)$ and is defined as* $f\left(A^t\right) = \left\{y, \mu_{f\left(A^t\right)}\left(y\right), \nu_{f\left(A^t\right)}\left(y\right) \mid y \in Y\right\}$, *where*

$$\mu_{f\left(A^t\right)}\left(y\right) = \begin{cases} \sup\limits_{x \in f^{-1}\left(y\right)} \mu_{A^t}\left(x\right) & \text{if} \quad f^{-1}\left(y\right) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{f\left(A^t\right)}\left(y\right) = \begin{cases} \inf\limits_{x \in f^{-1}\left(y\right)} \nu_{A^t}\left(x\right) & \text{if} \quad f^{-1}\left(y\right) \neq \varnothing \\ 1 & \text{otherwise} \end{cases}$$

Also, the inverse image of B^t under f is denoted by $f^{-1}\left(B^t\right)$ and is defined as $f^{-1}\left(B^t\right)\left(x\right) = \left\{x, \mu_{f^{-1}\left(B^t\right)}\left(x\right), \nu_{f^{-1}\left(B\right)}\left(x\right) \mid x \in X\right\}$, where $\mu_{f^{-1}\left(B^t\right)}\left(x\right) = \mu_{B^t}\left(f\left(x\right)\right)$ and $\nu_{f^{-1}\left(B^t\right)})\left(x\right) = \nu_{B^t}\left(f\left(x\right)\right)$ for all $x \in X$ and $t \in \left[0,1\right]$.

Note: For any $x \in X$, we have $\mu_{f\left(A^t\right)}\left(f\left(x\right)\right) \geq \mu_{A^t}\left(x\right)$ and $\nu_{f\left(A^t\right)}\left(f\left(x\right)\right) \leq \nu_{A^t}\left(x\right)$.

Theorem 4.2 *Let f: $X \to Y$ be an epimorphism of PMS-algebras and $t \in \left[0,1\right]$. If A^t is a t-IF PMS-SA of X, then $f\left(A^t\right)$ is a t-IF PMS-SA of Y.*

Proof. Let $y_1, y_2 \in Y$. Since $f: X \to Y$ is an epimorphism of PMS-algebras, there exist some $x_1, x_2 \in X$ such that $f\left(x_1\right) = y_1$ and $f\left(x_2\right) = y_2$. So using Definiton 3.1 and Definition 4.1 along with the definition of hommomrphism we have

$$\mu_{(f(A))^t}\left(y_1 * y_2\right) = \mu_{f\left(A^t\right)}\left(y_1 * y_2\right)$$

$$= \mu_{f\left(A^t\right)}\left(f\left(x_1\right) * f\left(x_2\right)\right) \left(\text{By Remark2.3}\right)$$

$$= \mu_{f\left(A^t\right)}\left(f\left(x_1 * x_2\right)\right)$$

$$\geq \mu_{\mathrm{A}}\left(x_1 * x_2\right)$$

$$\geq min\left\{\mu_{A^t}\left(x_1\right), \mu_{A^t}\left(x_2\right)\right\}, \forall x_1, x_2 \in X \,\text{suchthat}\, f\left(x_1\right) = y_1 \,\text{and}\, f\left(x_2\right) = y_2$$

$$= min\left\{\mu_{f\left(A^t\right)}\left(f\left(x_1\right)\right), \mu_{f\left(A^t\right)}\left(f\left(x_2\right)\right)\right\}$$

$$= min\left\{\mu_{f\left(A^t\right)}\left(y_1\right), \mu_{f\left(A^t\right)}\left(y_2\right)\right\}$$
$$= min\left\{\mu_{\left(f\left(A\right)\right)^t}\left(y_1\right), \mu_{\left(f\left(A\right)\right)^t}\left(y_2\right)\right\}$$

and

$$\nu_{\left(f\left(A\right)\right)^t}\left(y_1 * y_2\right) = \nu_{f\left(A^t\right)}\left(y_1 * y_2\right)$$
$$= \nu_{f\left(A^t\right)}\left(f\left(x_1\right) * f\left(x_2\right)\right) \left(\text{By Remark2.3}\right)$$
$$= \nu_{f\left(A^t\right)}\left(f\left(x_1 * x_2\right)\right)$$
$$\leq \nu_{\mathrm{A}}\left(x_1 * x_2\right)$$
$$\leq max\left\{\nu_{A^t}\left(x_1\right), \nu_{A^t}\left(x_2\right)\right\}, \forall x_1, x_2 \in X \,\text{suchthat}\, f\left(x_1\right) = y_1 \,\text{and}\, f\left(x_2\right) = y_2$$
$$= max\left\{\nu_{f\left(A^t\right)}\left(f\left(x_1\right)\right), \nu_{f\left(A^t\right)}\left(f\left(x_2\right)\right)\right\}$$
$$= max\left\{\nu_{f\left(A^t\right)}\left(y_1\right), \nu_{f\left(A^t\right)}\left(y_2\right)\right\}$$
$$= max\left\{\nu_{\left(f\left(A\right)\right)^t}\left(y_1\right), \nu_{\left(f\left(A\right)\right)^t}\left(y_2\right)\right\}$$

Hence $f\left(A^t\right)$ is a t-IF PMS-SA of Y.

Theorem 4.3 *Let* $f : X \to Y$ *be a homomorphism of PMS-algebras and* $t \in \left[0,1\right]$. *If* B^t *is a t-IF PMS-SA of* Y, *then* $f^{-1}\left(B^t\right)$ *is a t-IF PMS-SA of* X.

Proof. Let B^t be a t-intuitionistic fuzzy PMS-subalgebra of Y for $t \in \left[0,1\right]$. Let $x, y \in X$. Then By Definitin 3.1 and Definitin 4.1, we have

$\mu f{-}1(Bt)(x{*}y){=}\mu Bt(f(x{*}y)) = \mu Bt(f(x){*}f(y))$
$3min\{\mu Bt(f(x)), \mu Bt^{(} f(y))\}$
$=min\{\mu f{-}1(Bt)(x), \mu f{-}1(Bt)(y)\}$

And

$\nu f{-}1(Bt)(x{*}y){=}\nu Bt(f(x{*}y)) = \nu Bt(f(x){*}f(y))$

£$max\{\nu Bt(f(x)), \nu Bt(f(y))\}$

$=max\{\nu f{-}1(Bt)(x), \nu f{-}1(Bt)(y)\}$
$f^{-1}\left(B^t\right)$ is a t-IF PMS-SA of X.

The Converse of the above theorem is true if f is an epimorphism of PMS-algebras.

Theorem 4.4 Let $f: X \to Y$ be an epimorphism of PMS-algebras and B^t is a t-IFS in Y. If $f^{-1}\left(B^t\right)$ **is a t-IF** *PMS*-SA *of* X *, then* B^t *is a t-IF PMS-SA of* Y *for* $t \in \left[0,1\right]$.

Proof. Assume that $f: X \to Y$ *is an epimorphism of PMS-algebras and* $f^{-1}\left(B^t\right)$ *is a t*-IF *PMS-S*A of X. Let $y_1, y_2 \in Y$. Since f is an epimorphism of PMS-algebras, there exist some $x_1, x_2 \in X$ such that $f\left(x_1\right) = y_1$ and $f\left(x_2\right) = y_2$.

Now, $\mu_{B^t}\left(y_1 * y_2\right) = \mu_{B^t}(f\left(x_1\right) * f\left(x_2\right)$

$= \mu_{B^t}\left(f\left(x_1 * x_2\right)\right)$

$= \mu_{f^{-1}\left(B^t\right)}\left(x_1 * x_2\right)$

$\geq \min\left\{\mu_{f^{-1}\left(B^t\right)}\left(x_1\right), \mu_{f^{-1}\left(B^t\right)}\left(x_2\right)\right\}$

$= min\left\{\mu_{B^t}\left(f\left(x_1\right)\right), \mu_{B^t}\left(f\left(x_2\right)\right)\right\}$

$= min\left\{\mu_{B^t}\left(y_1\right), \mu_{B^t}\left(y_2\right)\right\}$

and

$\nu_{B^t}\left(y_1 * y_2\right) = \nu_{B^t}(f\left(x_1\right) * f\left(x_2\right)$

$= \nu_{B^t}\left(f\left(x_1 * x_2\right)\right)$

$= \nu_{f^{-1}\left(B^t\right)}\left(x_1 * x_2\right)$

$\leq \max\left\{\nu_{f^{-1}\left(B^t\right)}\left(x_1\right), \nu_{f^{-1}\left(B^t\right)}\left(x_2\right)\right\}$

$= max\left\{\nu_{B^t}\left(f\left(x_1\right)\right), \nu_{B^t}\left(f\left(x_2\right)\right)\right\}$

$= max\left\{\nu_{B^t}\left(y_1\right), \nu_{B^t}\left(y_2\right)\right\}$

Hence B^t is a is a t-IF PMS-SA of Y.

Lemma 4.5 Let $f: X \to Y$ be an epimorphism **of PMS-***algebras and* A^t *be an IF PMS-SA of* X *, then* $f\left(C_{\alpha,\beta}\left(A^t\right)\right) \subseteq C_{\alpha,\beta}\left(f\left(A^t\right)\right)$

Proof. Let $y \in f\left(C_{\alpha,\beta}\left(A^t\right)\right)$*. Since* f *is a*n *epimor*phism of PMS-algebras, there exist $x \in C_{\alpha,\beta}\left(A^t\right))$ such that $f\left(x\right) = y$. Then $\mu_{A^t}\left(x\right) \geq \alpha$ and $\nu_{A^t}\left(x\right) \leq \beta$. By definition 4.1 we have $\mu_{f\left(A^t\right)}\left(y\right) = sup\left\{\mu_{A^t}\left(x\right) : x \in f^{-1}\left(y\right)\right\} \geq \mu_{A^t}\left(x\right) \geq \alpha$ and

$\nu_{f\left(A^t\right)}\left(y\right) = inf\left\{\mu_{A^t}\left(x\right) : x \in f^{-1}\left(y\right)\right\} \leq \mu_{A^t}\left(x\right) \leq \beta$

$\Rightarrow \mu_{f\left(A^t\right)}\left(y\right) \geq \alpha$ and $\nu_{f\left(A^t\right)}\left(y\right) \leq \beta \Rightarrow y \in C_{\alpha,\beta}\left(f\left(A^t\right)\right)$.

Therefore $f\left(C_{\alpha,\beta}\left(A^t\right)\right) \subseteq C_{\alpha,\beta}\left(f\left(A^t\right)\right)$.

Theorem 4.6 **Let** A^t **be a** t *-IFS of a PMS-algebra* X *and* $f: X \to Y$ *be an e*pimorphism *of PMS-algebras. Then the homomorphic image of the nonempty subset* $C_{\alpha,\beta}\left(A^t\right)$ *of X is a PMS-SA of Y.*

Proof. *Let* A^t be a t-IF PMS-SA of X and let $y_1, y_2 \in Y$. Since f is an epimorphism of PMS-algebras, there exist $x_1, x_2 \in X$ such that $f\left(x_1\right) = y_1$ and $f\left(x_2\right) = y_2$. By theorem 4.2 $f\left(A^t\right)$ is a t-IF PMS-SA of Y and By Theorem 3.15 $C_{\alpha,\beta}\left(A^t\right)$ is a PMS-SA of X.

So that $x_1 * x_2 \in C_{\alpha,\beta}\left(A^t\right)$ for each $x_1, x_2 \in C_{\alpha,\beta}\left(A^t\right)$. Now let $y_1, y_2 \in f\left(C_{\alpha,\beta}\left(A^t\right)\right)$.

By Lemma 4.5 $y_1, y_2 \in C_{\alpha,\beta}\left(f\left(A^t\right)\right)$

$$\Rightarrow \mu_{f\left(A^t\right)}\left(y_1\right) \geq \alpha,\ \mu_{f\left(A^t\right)}\left(y_2\right) \geq \alpha \text{ and } \nu_{f\left(A^t\right)}\left(y_1\right) \leq \beta,\ \nu_{f\left(A^t\right)}\left(y_2\right) \leq \beta.$$

Now, $\mu_{(f(A))^t}\left(y_1 * y_2\right) = \mu_{f\left(A^t\right)}\left(y_1 * y_2\right)$

$$= \mu_{f\left(A^t\right)}\left(f\left(x_1\right) * f\left(x_2\right)\right)$$

$$= \mu_{f\left(A^t\right)}\left(f\left(x_1 * x_2\right)\right)$$

$$\geq \mu_{A^t}\left(x_1 * x_2\right) \geq \alpha$$

and

$$\nu_{(f(A))^t}\left(y_1 * y_2\right) = \nu_{f\left(A^t\right)}\left(y_1 * y_2\right)$$

$$= \nu_{f\left(A^t\right)}\left(f\left(x_1\right) * f\left(x_2\right)\right)$$

$$= \nu_{f\left(A^t\right)}\left(f\left(x_1 * x_2\right)\right)$$

$$\leq \nu_{A^t}\left(x_1 * x_2\right) \leq \beta$$

Thus $\mu_{(f(A))^t}\left(f\left(x_1\right) * f\left(x_2\right)\right) = \mu_{(f(A))^t}\left(y_1 * y_2\right) \geq \alpha$ and

$$\nu_{(f(A))^t}\left(f\left(x_1\right) * f\left(x_2\right)\right) = \nu_{(f(A))^t}\left(y_1 * y_2\right) \leq \beta$$

Threrefore $y_1 * y_2 = f\left(x_1\right) * f\left(x_2\right) \in f\left(C_{\alpha,\beta}\left(A^t\right)\right)$. Hence $f\left(C_{\alpha,\beta}\left(A^t\right)\right)$ is a PMS-SA of Y.

Lemma 4.7 *Let* $f: X \to Y$ *be a homomorphism of PMS-algebras and* B^t *be an IF PMS-SA of* Y, *then* $f^{-1}\left(C_{\alpha,\beta}\left(B^t\right)\right) = C_{\alpha,\beta}\left(f^{-1}\left(B^t\right)\right)$

Theorem 4.8 *Let* B^t *be a* t*-IFS of a PMS-algebra* Y *and* $f: X \to Y$ *be an epimorphism of PMS-algebras. Then the homomorphic inverse image of the nonempty subset* $C_{\alpha,\beta}\left(B^t\right)$ *of Y is a PMS-SA of X.*

Proof. Let B^t be a t-IF PMS-SA of Y and let $y_1, y_2 \in Y$. Since f is an epimorphism of PMS-algebras, there exist $x_1, x_2 \in X$ such that $f\left(x_1\right) = y_1$ and $f\left(x_2\right) = y_2$. By theorem 4.3 $f^{-1}\left(B\right)$ is a

t-IF PMS-SA of Y and By Theorem 3.15 $C_{\alpha,\beta}\left(B^t\right)$ is a PMS-SA of Y. Assume that $y_1, y_2 \in C_{\alpha,\beta}\left(B^t\right)$. Then $y_1 * y_2 \in C_{\alpha,\beta}\left(B^t\right)$. So, we have

$$\mu_{f^{-1}\left(B^t\right)}\left(x_1 * x_2\right) = \mu_{B^t}\left(f\left(x_1 * x_2\right)\right)$$
$$= \mu_{B^t}\left(f\left(x_1\right) * f\left(x_2\right)\right)$$
$$= \mu_{B^t}\left(y_1 * y_2\right) \geq \alpha$$

and

$$\nu_{f^{-1}\left(B^t\right)}\left(x_1 * x_2\right) = \nu_{B^t}\left(f\left(x_1 * x_2\right)\right)$$
$$= \nu_{B^t}\left(f\left(x_1\right) * f\left(x_2\right)\right)$$
$$= \nu_{B^t}\left(y_1 * y_2\right) \leq \beta$$

$\Rightarrow \mu_{f^{-1}\left(B^t\right)}\left(x_1 * x_2\right) \geq \alpha$ and $\nu_{f^{-1}\left(B^t\right)}\left(x_1 * x_2\right) \leq \beta$. Therefore $x_1 * x_2 \in C_{\alpha,\beta}\left(f^{-1}\left(B^t\right)\right)$.

Since $C_{\alpha,\beta}\left(f^{-1}\left(B^t\right)\right) = f^{-1}\left(C_{\alpha,\beta}\left(B^t\right)\right)$ By Lemma 4.7 it follows that $f^{-1}\left(C_{\alpha,\beta}\left(B^t\right)\right)$ is a PMS-SA of X.

5. CARTESIAN PRODUCT OF t-INTUITIONISTIC FUZZY PMS-SUBALGEBRAS

In this section, we introduce the concept of Cartesian product of a t-intuitionistic fuzzy PMS-subalgebras and show that the Cartesian product of any two t-IF PMS-SAs of a PMS-algebra forms a t-IF PMS-SA.

Theorem 5.1 *Let A^t and B^t be two t-IF PMS-SAs of X and Y respectively. Then $A^t \times B^t$ is a t-IF PMS-SA of $X \times Y$.*

Proof. Let $\left(x_1, y_1\right), \left(x_2, y_2\right) \in X \times Y$ and $t \in \left[0,1\right]$. Then by Definition 3.1

$\mu At{\times}Bt((x1,y1)*(x2,y2)) = \mu At{\times}Bt(x1*x2,y1*y2)$
$= min\{\mu At(x1*x2),\mu Bt(y1*y2)\}$
$3min\{min\{\mu At(x1),\mu At(x2)\},min^{\{}\mu Bt(y1),\mu Bt(y2)\}\}$
$= min\{min\{\mu At(x1),\mu Bt(y1)\},min\{\mu At(x2),\mu Bt(y2)\}\}$
$= min\{\mu At{\times}Bt(x1,y1),\mu At{\times}Bt(x2,y2)\}$
And
$\nu At{\times}Bt((x1,y1)*(x2,y2)) = \nu At{\times}Bt(x1*x2,y1*y2)$

$= max\{\nu At(x1*x2),\nu Bt(y1*y2)\}$

£$max\{max\{\nu At(x1),\nu At(x2)\},max\{\nu Bt(y1),\nu Bt(y2)\}\}$
$= max\{max\{\nu At(x1),\nu Bt(y1)\},max\{\nu At(x2),\nu Bt(y2)\}\}$
$= max\{\nu At{\times}Bt(x1,y1),\nu At{\times}Bt(x2,y2)\}$

Hence $A^t \times B^t$ is a t-IF PMS-SA of $X \times Y$.

Theorem 5.2 Let A^t and B^t be any two t-IF PMS-SAs of X and Y respectively. Then $\Box\left(A^t \times B^t\right)$ is also a t-IF PMS-SA of $X \times Y$.

Proof. Since A^t and B^t are t-IF PMS-SAs of X and Y respectively, then by **Theorem 5.1** $A^t \times B^t$ *is a* t*-IF PMS-SA of* $X \times Y$. *Thus, for all* $\left(x_1, y_1\right), \left(x_2, y_2\right) \in X \times Y$

$$\mu_{A^t \times B^t}\left(\left(x_1, y_1\right) * \left(x_2, y_2\right)\right) \geq min\left\{\mu_{A^t \times B^t}\left(x_1, y_1\right), \mu_{A^t \times B^t}\left(x_2, y_2\right)\right\}$$

So, to prove $\Box\left(A^t \times B^t\right)$ is *a* t*-IF PMS-SA of* $X \times Y$, *it suffices* to show that

$\bar{\mu}_{A^t \times B^t}(\left(x_1, y_1\right) * \left(x_2, y_2\right) \leq min\left\{\bar{\mu}_{A^t \times B^t}\left(x_1, y_1\right), \bar{\mu}_{A^t \times B^t}\left(x_2, y_2\right)\right\}$,for $\left(x_1, y_1\right), \left(x_2, y_2\right) \in X \times Y$.Now let $\left(x_1, y_1\right), \left(x_2, y_2\right) \in X \times Y$, then

$$\bar{\mu}_{A^t \times B^t}\left(\left(x_1, y_1\right) * \left(x_2, y_2\right)\right) = 1 - \mu_{A^t \times B^t}\left(\left(x_1, y_1\right) * \left(x_2, y_2\right)\right)$$

$$\leq 1 - min\left\{\mu_{A^t \times B^t}\left(x_1, y_1\right), \mu_{A^t \times B^t}\left(x_2, y_2\right)\right\}$$

$$= max\left\{1 - \mu_{A^t \times B^t}\left(x_1, y_1\right), 1 - \mu_{A^t \times B^t}\left(x_2, y_2\right)\right\}$$

$$= max\left\{\bar{\mu}_{A^t \times B^t}\left(\left(x_1, y_1\right), \bar{\mu}_{A^t \times B^t}\left(x_2, y_2\right)\right)\right\}$$

$$\Rightarrow \bar{\mu}_{A^t \times B^t}\left(\left(x_1, y_1\right) * \left(x_2, y_2\right)\right) \leq max\left\{\bar{\mu}_{A^t \times B^t}\left(x_1, y_1\right), \bar{\mu}_{A^t \times B^t}\left(x_2, y_2\right)\right\}$$

Hence $\Box\left(A^t \times B^t\right)$ is a t-IF PMS-SA of $X \times Y$.

Theorem 5.3 Let A^t and B^t be any two t-IF PMS-SAs of X and Y respectively. Then $\Diamond\left(A^t \times B^t\right)$ is also a t-IF PMS-SA of $X \times Y$.

Proof. By Theorem 5.1 $A^t \times B^t$ is **a** t **-IF PMS**-*SA of* $X \times Y$. S*o, to prove* $\Diamond\left(A^t \times B^t\right)$ *is a* t *-IF PMS-SA* of *of* $X \times Y$, *it suffices to show that*

$$\bar{\nu}_{A^t \times B^t}(\left(x_1, y_1\right) * \left(x_2, y_2\right) \geq min\left\{\bar{\nu}_{A^t \times B^t}\left(x_1, y_1\right), \bar{\nu}_{A^t \times B^t}\left(x_2, y_2\right)\right\}.$$

*Now l*et $\left(x_1, y_1\right), \left(x_2, y_2\right) \in X \times Y$, *the*n

Definition 5.4 Let A^t and B^t be t-IFSs of X and Y w.r.t IFSs A and B. Then the $\left(\alpha, \beta\right)$-cut of $A^t \times B^t$ is a crisp subset $C_{\left(\alpha,\beta\right)}\left(A^t \times B^t\right)$ of $X \times Y$ **given by**

$C_{\left(\alpha,\beta\right)}\left(A^t \times B^t\right) = \left\{\left(x, y\right) \in X \times Y : \mu_{A^t}\left(x, y\right) \geq \alpha,\ \nu_{A^t}\left(x, y\right) \leq \beta\right\}$**, w**here $\alpha, \beta \in \left[0,1\right]$ with $\alpha + \beta \leq 1$.

Theorem 5.5 Let A^t and B^t be t-IFSs of X and Y reapectively. Then $A^t \times B^t$ is a t-IF PMS-SA of $X \times Y$ if and only **if the non**empty subset $C_{\left(\alpha,\beta\right)}\left(A^t \times B^t\right)$ of $X \times Y$ is a PMS-SA of $X \times Y$ for all $\alpha, \beta \in \left[0,1\right]$ with $\alpha + \beta \leq 1$.

Proof. Let $A^t = \left(\mu_{A^t}, \nu_{A^t}\right)$ and $B^t = \left(\mu_{B^t}, \nu_{B^t}\right)$ be t-IFSs of X and Y respectively. Let $\left(x_1, y_1\right), \left(x_2, y_2\right) \in X \times Y$ such that $\left(x_1, y_1\right), \left(x_2, y_2\right) \in C_{\left(\alpha,\beta\right)}\left(A^t \times B^t\right)$ for $\alpha, \beta \in \left[0,1\right]$. Then $\mu_{A^t \times B^t}\left(x_1, y_1\right) \geq \alpha,\ \mu_{A^t \times B^t}\left(x_2, y_2\right) \geq \alpha$ and $\nu_{A^t \times B^t}\left(x_1, y_1\right) \leq \beta,\ \nu_{A^t \times B^t}\left(x_2, y_2\right) \leq \beta$. Since $A^t \times B^t$ is a t-IF PMS-SA of $X \times Y$, we have

$\mu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)\geq min\left\{\mu_{A^t\times B^t}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}\geq min\left\{\alpha,\alpha\right\}=\alpha$ and
$\nu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)\leq max\left\{\mu_{A^t\times B^t}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}\leq max\left\{\beta,\beta\right\}=\beta$.

$\Rightarrow \mu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)\geq\alpha$ and $\nu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)\leq\alpha$.

Therefore, $\left(x_1,y_1\right)*\left(x_2,y_2\right)\in C_{(\alpha,\beta)}\left(A^t\times B^t\right)$.

Hence $C_{(\alpha,\beta)}\left(A^t\times B^t\right)$ is a PMS-subalgebra of $X\times Y$.

Conversely, Suppose $C_{(\alpha,\beta)}\left(A^t\times B^t\right)$ is a PMS-subalgebra of $X\times Y$ for all $\alpha,\ \beta\in\left[0,1\right]$ with $\alpha+\beta\leq 1$. Assume that $A^t\times B^t$ is not a t-IF PMS-SA of $X\times Y$. Then there exists $\left(x_1,y_1\right),\left(x_2,y_2\right)\in X\times Y$ such that

$\mu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)< min\left\{\mu_{A\times B}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}$ and
$\nu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)> max\left\{\mu_{A^t\times B^t}\left(x_1,y_y\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}$.

Then by taking $\alpha_0=\dfrac{1}{2}\left\{\mu_{A\times B}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)+min\left\{\mu_{A\times B}\left(x_1,y_1\right),\mu_{A\times B}\left(x_2,y_2\right)\right\}\right\}$ and
$\beta_0=\dfrac{1}{2}\left\{\nu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)+max\left\{\nu_{A^t\times B^t}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}\right\}$
we get $\mu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)<\alpha_0<min\left\{\mu_{A^t\times B^t}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}$ and

$\nu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)>\beta_0>max\left\{\mu_{A^t\times B^t}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}$.

Hence, $\left(x_1,y_1\right)*\left(x_2,y_2\right)\notin C_{(\alpha,\beta)}\left(A^t\times B^t\right)$ but $\left(x_1,y_1\right)\in C_{(\alpha,\beta)}\left(A^t\times B^t\right)$ and $\left(x_2,y_2\right)\in C_{(\alpha,\beta)}\left(A^t\times B^t\right)$. This implies $C_{(\alpha,\beta)}\left(A^t\times B^t\right)$ is not a PMS-subalgebra of $X\times Y$, which is a contradiction.

Therefore $\mu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)\geq min\left\{\mu_{A^t\times B^t}\left(x_1,y_1\right),\mu_{A^t\times B^t}\left(x_2,y_2\right)\right\}$ and
$\nu_{A^t\times B^t}\left(\left(x_1,y_1\right)*\left(x_2,y_2\right)\right)\leq max\left\{\nu_{A^t\times B^t}\left(x_1,y_1\right),\nu_{A^t\times B^t}\left(x_2,y_2\right)\right\},\forall\left(x_1,y_1\right),\left(x_2,y_2\right)\in X\times Y$.

Hence $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$.

6. CONCLUSION

In the present article, we applied the concept of a t-intuitionistic fuzzy set to PMS-subalgebras of PMS-algebras. We introduced the concept of t-intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and investigated associated properties. We proved that every intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is a t-intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and showed that the converse is not necessarily true by example. We established a condition for an intuitionistic fuzzy set in a PMS-algebra to be a t-intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. We described the t-intuitionistic fuzzy PMS-subalgebras of PMS-algebra by their (α, β) level cuts. We investigated that the homomorphic images and inverse images of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra are also the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra. Moreover, we proved that the homomorphic images and inverse images of the nonempty $\left(\alpha,\beta\right)$ cuts of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra are again PMS-subalgebras of a PMS-algebra. Finally, we demonstrated that the Cartesian product of the t-intuitionistic fuzzy PMS-subalgebras of a PMS-algebra is also the t-intuitionistic fuzzy PMS-subalgebra and characterized it in terms of its $\left(\alpha,\beta\right)$ level cuts. We believe that the results of this study will add new dimensions to the structures of PMS-algebras based on t-intuitionistic fuzzy sets and serve as a foundation for future studies in the fuzzification of PMS-algebra. To gain additional novel results, we will extend this notion to t-Q intuitionistic fuzzy PMS-subalgebras, t-intuitionistic multi-fuzzy, and anti-multi-fuzzy PMS-subalgebras in our future works. Furthermore, we will develop the Neuro-fuzzy algebraic structures for PMS-subalgebras of PMS-algebras.

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CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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