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Observations of Chaotic Behaviour in Nonlinear Inventory Models

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ABSTRACT

This article describes the use of simulation to investigate incipient chaotic behaviour in inventory models. Model structures investigated were either capacity limited or of variable delay time, implemented in discrete and continuous transform algebras. Results indicate the absence of chaos for a continuous time model but gave limited evidence for chaos in both unrestricted discrete models and those with a positive orders only limit. The responses where interaction with the capacity limit occurred did not confirm chaotic behaviour at odds with published results. Using the Liapunov exponent as a measure of chaotic behaviour, the results indicated, where the delay varies in proportion to order rate, a larger fixed delay reduced the Liapunov exponent as did increasing the dependence of delay on order rate. The effect of the model structures showed that the IOBPCS model, produced the largest Liapunov exponent. Reducing the discrete model update time reduced the Liapunov exponent.

KEYWORDS

APVIOBPCS, Chaos, Continuous Simulation, Discrete Simulation, Inventory, Liapunov Exponents, Order Rate Capacity Limits, Variable Delay Models

INTRODUCTION

In modern engineering, science and management great use is made of models to enable predictions to be made. Studies of the dynamics of physical and human systems are usually based on experiments and on the decision processes used to control them. Such studies of the variation with time depend on the initial conditions and on the parameters to be determined, usually by experiment but also by theoretical analysis. Some deterministic dynamic systems have been shown to be subject to chaotic behavior (Drazin, 1992). Fawcett and Waller (2011) have shown clearly why rigorous theoretical analysis of business processes is as important to progress as more practical discussions. They argue for dual approach but with rigorous evaluation of all research. This paper attempts to answer the question whether evidence for chaos in supply chains is real.

Chaos has been defined by Wilding (1998) as "…aperiodic, bounded dynamics in a deterministic system with sensitivity dependence on initial conditions, and has structure in phase space…" Wilding further outlines these terms as:

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- **Aperiodic:** The same state is never repeated twice;
- **Bounded:** On successive iterations, the state stays within a finite range and does not approach ± infinity;
- **Deterministic:** There is a definite rule with no random terms governing the dynamics;
- **Sensitivity to initial conditions:** Two points that are initially close will drift apart as time proceeds;
- **Structure in phase space:** Nonlinear systems are described by multi-dimensional vectors. The space in which these vectors lie is called a phase space or state space. The dimensions of this phase space are an integer (Abarbanel 1996). Chaotic systems display discernible patterns when viewed.

Chaos is defined as a deterministic behavior of a system governed by fixed rules involving no random elements. The sensitivity to initial conditions is such that a minor change in any variable may result in a totally different response. Thompson and Stewart (2002) state that chaos is unpredictable over long time scales because any two phase space trajectories starting close to a chaotic attractor will separate as they progress in time. The separation rate will depend on the largest Liapunov exponent (Kapitaniak, 1998) that is related to the system eigenvalues. The phenomenon is referred to as Deterministic Chaos.

Chaotic behavior has been found in cardiac systems (Garfinkel et al, 1992), stock market performance (Weis, 1992) and management of telephone exchanges (Erramilli and Forys, 1991).

When a model is created of the chosen system experimental measurements may be used. The ultimate accuracy of experiments is necessarily limited both by time and money and by the character of the parameters themselves. It is therefore imperative to determine what effect small changes in value of parameters have on the overall system performance. This not only has a bearing on whether the system as described behaves as needed but to see if the model is a true representation of the real system i.e. to see how sensitive the system is to parameter changes. A review of the history and progress in understanding chaos is given by Thompson and Stewart (2002).

In section 2 a review is presented of the published evidence that chaotic behavior exists in supply chains. The original purpose of this research was to examine whether the real limits in an inventory model were the cause of chaos observed in the behavior of inventory systems. This is followed by an examination of various models of inventory of the APVIOBPCS type (Lalwani et al., 2006) to see if they incur chaotic behavior. The results of these simulations will show which models are susceptible to chaos and some of the features that allow chaotic states to exist. Some of the observations may allow a possible path manipulation to reduce the occurrence of chaos. Simulation was chosen for this work since the conditions are controllable and repeatable whereas observing real supply chains have the problem of not being able to measure all the parameters involved and simulated models may not include all the real relevant elements. In this work we do not include the costs as these are usually specific to different products and companies and their inclusion would introduce a range of limits and conditions.

In this paper, we examine three specific areas of inventory operation where nonlinearity exists or may be introduced:

1. The approximations introduced in the implementation of a discrete model;
2. The effects of resource capacity limits in a system;
3. The introduction of variable time delays that may depend on other variables in the system, such as order rate.

Since the APVIOBPCS inventory models are examined in this paper, we exclude the possible case of nonlinear inventory or WIP gains, which are to be examined elsewhere.

In the rest of the paper a review is made of reported chaos in supply chains, followed by a discussion of the inventory models used in this work. The numerical experiments made are divided into three groups; discrete models with no limits, those with order rate boundaries and those where the order rate affects the process delay time. In each case tests were made to eliminate numerical artefacts, simulations were tested for repeatability, integration method and the averaging process. The tests were very repeatable and internally consistent. A discussion is made of the implications of the results with conclusions and possible future work.

REVIEW OF REPORTS OF CHAOS IN SUPPLY CHAINS

A supply chain is a group of agents involved in supplying goods or services to a consumer, typically a factory supplies a distributor, who stocks a warehouse from which the retailers are supplied. Many of the management problems of these agents are concerned with maintaining their various individual inventory levels in the supply chain in response to demands, either minimizing it for lean operations or determining the correct value for agile operations. Riddalls and Bennett (2002) determined that inventory control behaves as an automatic pipeline and order-based production control system (APIOBPCS) which mimics the human behavior reported by Sterman (1989) from his observations of executives during playing the 'MIT Beer Game.' Mosekilde et al. (1988) used the Forrester model (1961) to illustrate possible chaotic modes of behavior in supply chains while Mosekilde and Larsen (1988) used results from the MIT beer game to illustrate limit cycle and deterministic chaos in the supply chain. They also found that the assumed ordering policy had a dominant effect on the system response. Pinder (1996) showed that demand for oil filters indicated considerable sensitivity to the original conditions with suspected chaotic behavior.

Wilding (1998) reviewed the implications of chaos for supply chain behavior using a simple spreadsheet to predict deterministic chaos, pointing out the existence of islands of stability in chaotic system time responses. Based on this work he outlined the difficulty of long-term planning and critically the lack of reproducibility of system behavior. All these effects are compounded by the use of ERP and other management software tools due to unjustifiable computer precision being incorporated into schedule calculations and thus generating very small parameter fluctuations in the system start-up conditions. Larsen et al. (1999) used the Beer game model for a 4-tier supply chain, to demonstrate; periodic, quasi-periodic, and chaotic and hyperchaotic motions in phase space. The paper supply industry was shown by Holmström and Hameri (1999) to have different dynamics for the mill level and the sales organization level. Kumara et al. (2003) used a queueing model to simulate a logistics system. They found clear evidence of chaos based on observations of bifurcation and the magnitude of the Liapunov exponents. However, the power spectra shown in their work are not typical of those of a chaotic system (Kapitaniak, 1998).

The work of Wang et al. (2005), studying sales of cars, electric drills and air conditioners in the Taiwanese market, found chaos and the authors devised a new production lot sizing algorithm, which gave very good results for a number of examples.

The beer game model was also used by Laugesan and Mosekilde (2006) to investigate boundary collisions (e.g. where the order rate cannot be negative or the order rate has a maximum value that cannot be exceeded value) in a 4-tier supply chain model of manufacturer, distributor, wholesaler and retailer. They investigated whether these collisions would result in bifurcation period doubling finding examples where the eigenvalues would jump after a border collision generating a Hopf bifurcation, (Hilborn, 1994), an effect which is recognized as clear evidence of chaos. Hwarng and Xie (2008) used the Beer game model to make a more detailed investigation of the effects of chaos amplification across the tiers of the supply chain. Hwarng and Yuan (2014) extended this work to the case of quasi-chaos when the input to the system was itself stochastic. They identified significant interaction between the supply chain system dynamics and the structure of the inputs to the system. Their analysis showing that extreme caution must be applied in making decisions about ordering policy. Yuan and Hwarng (2016) examine the stability and chaos in demand systems with the demand process influenced by customer backlash and social media information. They found that for an effective demand based strategic pricing policy social interactions have to be included. Recent developments have included attempts to devise control mechanisms for chaotic supply chains by Göksu et al. (2015). In this case they applied linear feedback with some success by using a Liapunov function (Liapunov, 1949) to achieve global stability.

INVENTORY MODELS

The work described here is based on the behavior of control engineering-based system models developed by Towill (1982) from the System Dynamics (SD) supply chain concepts of Forrester (1961). Since deterministic chaos behavior is to be investigated a step input in sales has been used to initiate the response.

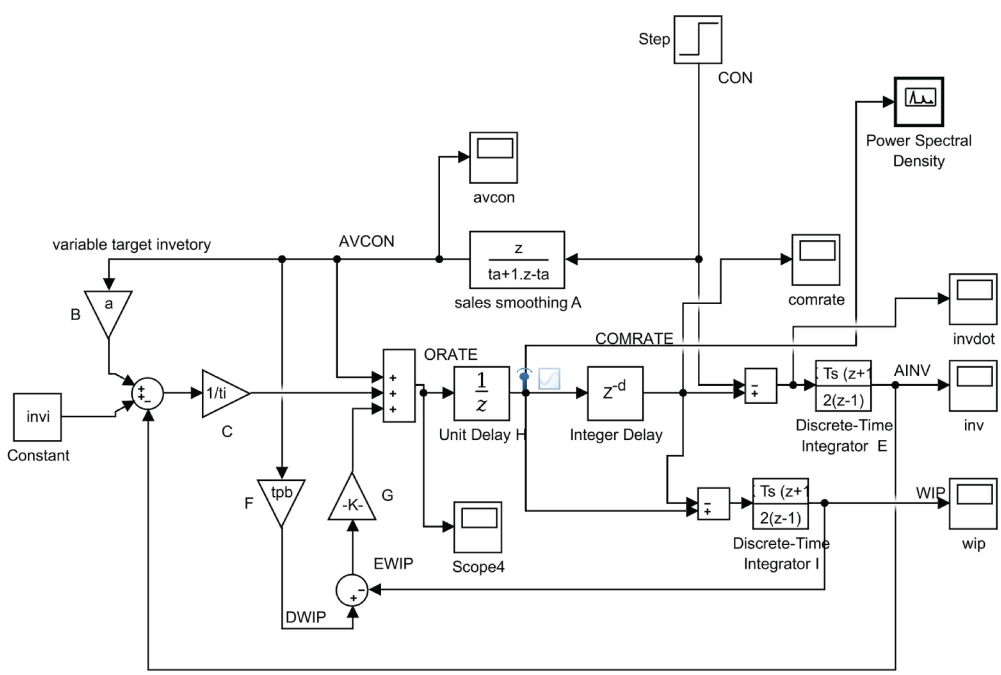
Disney and Towill (2002), Lalwani et al. (2006) at Cardiff University and White and Censlive (2017) have analyzed an Automatic Pipeline, Variable Inventory and Order Based Production Control Systems (APVIOBPCS) extensively, to determine its stability and optimization.

In an APVIOBPCS model of a factory and sales system (Figure 1) the sales orders are modified by the factory using a forecasting system (modelled by an effective exponential delay in this representation). These modified sales demands are added to a fraction of the inventory error, the difference between current inventory and the target inventory, plus a fraction of the Work in Progress (WIP) error to produce an order rate. Which, after a single delay representing the totality of manufacturing processes, will lead to production being completed. From this completion rate the sales rate is subtracted and the accumulation of these differences leads to the inventory of products.

The results of Sterman (2000) show that when capacity limits are introduced in the supply process the delayed feedback to the human scheduler causes quite oscillatory behavior. Wilding (1998) gives a general description of the effects chaotic responses might have on supply chains.

Delays due to production, for example, are described by a simple single time constant *tp*. The key to the way the inventory system behaves is the rate of ordering. The inventory error (EINV) is

Figure 1. Simulink model of discrete APVIOBPCS without limit



defined as the difference between a desired level of inventory (TINV) added to a variable inventory target (a*AVCON) and the actual inventory (AINV). An exponential averaging function is used to obtain the mean sales consumption (AVCON) as a function of the sales orders (CONS). This is used to obtain the order rate (ORATE) given to the production facility, wherever it is and whoever controls it. There is a delay in the production process. Disney and colleagues (2002) represented this production delay as a discrete function although it was originally implemented by Forrester (1961) by an exponential function of the form e^{-st}. The stock-out problem is a direct result of this factory delay lagging sales demands.

Analytical models used here are described using Laplace Transforms as well as z transforms. System responses are analyzed using two types of delay representation; an exponential delay function and a finite delay. The sales smoothing computation, which was represented by an exponential form, is a good approximation to business practice, although several other smoothing functions have been proposed (Synetos et al., 2009). These have not been explored as widely by modelers.

Model Type

Inventory systems described by equation 1 can be modelled either as a continuous system, where the events are represented by differential equations such as:

$$AINV\left(t\right) = \int_0^t \left(COMRATE\left(t\right) - CONS\left(t\right)\right)dt \tag{1}$$

and then expressed in Laplace transform notation or as a discrete time system (equations 2-equation 12) where the events are evaluated at a set time interval as in a microprocessor clock system and represented by difference equations (Åström & Wittenmark, 1997):

$$AINV_t = AINV_{t-1} + COMRATE_t - CONS_t \tag{2}$$

In this later case:

$$t = kT \tag{3}$$

where T is a fixed time period corresponding to a periodic review of information, in this work the time increment was taken to be typically one week but the effect of changing this time is considered later where k is an integer. The effect of changes to T is shown in Figure 14. The equations representing the system using the difference equation format are given below:

- Consumption forecast:

$$AVCON_t = AVCON_{t-1} + \frac{1}{T + ta}\left(CONS_t - AVCON_{t-1}\right) \tag{4}$$

- Target or desired Work In Progress (WIP):

$$DWIP_t = AVCON_t t\overline{p} \tag{5}$$

- Actual WIP:

$$WIP_t = WIP_{t-1} + ORATE_t - COMRATE_t \tag{6}$$

- Inventory error:

$$EINV_t = TINV_t - AINV_t + aAVCON_t \tag{7}$$

- Order rate:

$$ORATE_t = AVCON_{t-1} + \frac{\left(EINV_{t-1}\right)}{ti} + \frac{\left(DWIP_{t-1} - WIP_{t-1}\right)}{tw} \tag{8}$$

- Completion rate for a discrete delay *tp,* is given by:

$$COMRATE_t = ORATE_{t-tp} \tag{9}$$

- Error in WIP:

$$EWIP_t = DWIP_t - WIP_t = t_{\bar{p}} AVCON_t - WIP_t \tag{10}$$

Typical test input $CONS_t = \begin{Bmatrix} 0 \, if \, t \le 0 \\ 1 \, if \, t > 0 \end{Bmatrix}$ for a step input (11)

Typical target inventory $TINV_t = 0$ (12)

The discrete model is shown in Figure 1 as a Simulink model. The input sales demand is a step function which is fed via a smoothing function to the variable target inventory and to the WIP demand after multiplication by the estimated delay. The order rate is computed from the inventory error term, a measure of the WIP error and the smoothed demand. The order rate is delayed and has the sales subtracted. After integration this leads to the Actual inventory value (AINV).

The work described in this paper was simulated using Simulink® and MATLAB® version R2016b. Simulink® is a general simulation tool that can simulate both differential and difference equations.

Delay Representation

There are four possible combinations of APVIOBPCS model and delay implementations (Towill, 1982; Disney & Towill, 2002; White & Censlive, 2017):

1. Discrete models with a finite delay;
2. Discrete models with an exponential delay;
3. Continuous models with a finite delay;
4. Continuous models with an exponential delay.

These represent the extremes of the model, the exponential model is a smooth transition and the finite delay a sudden sharp transition.

In the case 1 the delay is represented by a function of the form z^{-tp} whereas in case 2 the delay is represented by the function:

$$\frac{Tz}{\left(ta+T\right)z-ta}$$

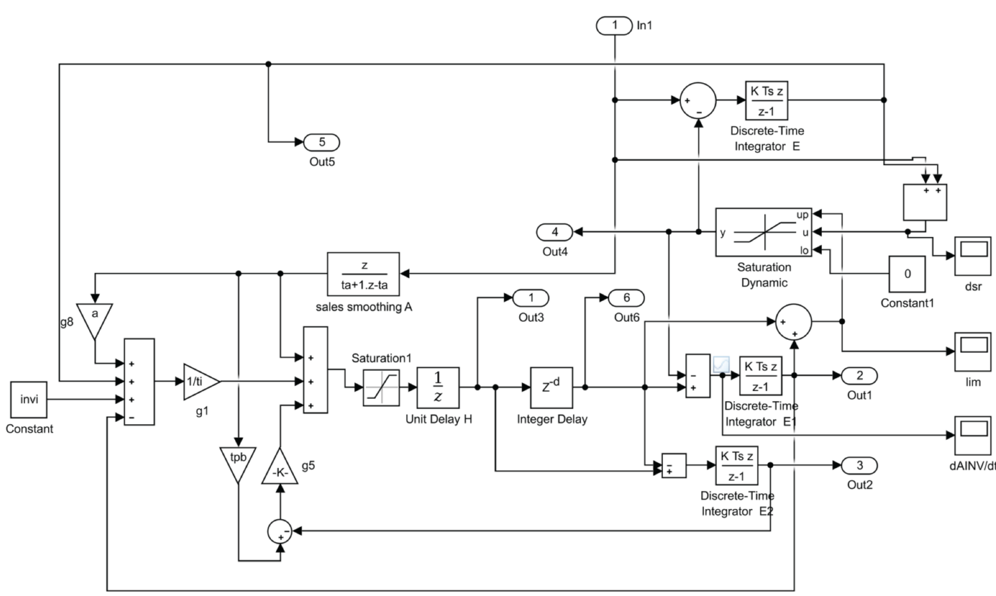
This form uses using backwards Euler differences. For complete rigor tests of several different approximate implementations of z transform of the delay were modelled showed slightly different results but not a significant factor to the overall performance of the models. For the two other models the finite delay in a continuous model is represented by e^{-stp} and in the exponential delay by $\frac{1}{tps+1}$. The system response in case 4 is never unstable and does not lead to chaos or quasi periodic motion if *ti* and *tw* are positive.

Nonlinear Models

Three distinct situations are examined in this paper. First the case of a discrete model with no limits on orate i.e. returns are allowed. The second case considered is that where the production or process capacity is limited, here the nonlinear inventory model was derived from Spiegler (2013) (Figure 2) with the constraint that the order rate could not be negative i.e. no returns allowed. This is implemented in the models by two saturation functions.

The third case is where there are no order rate limits but the production delay is variable and depends on order rate such as would occur if the production were varied to meet increased demand. Several authors have considered how this dependence can be modelled. Anli et al. (2007) suggest dependency on lot size, production mix, schedule and other operational practices. Chen and Chang

Figure 2. Simulink model of APVIOBPCS with order limits



(2007) produced optimal responses for inventory with resource constraints and variable lead times while Chandra and Grabis (2008) examined how the delay time depended on costs. The relationship between price and delay times was found using stochastic hypotheses developed by Wijaya and Purwanto (2013) who proposed, albeit on a small sample based on observations of steel production that the connection was "altogether that price, lead time and delay have a relationship to order quantity".

A simple first approximation is to assume that the delay time depends on the order rate alone and the basic representation of this dependence is a linear function of the form:

$$tp = tp_0 + orate * tp_1 \tag{13}$$

The simulation model is shown in Figure 3. Here the variable delay is computed with a variable delay block and an initial value plus an order rate dependent quantity.

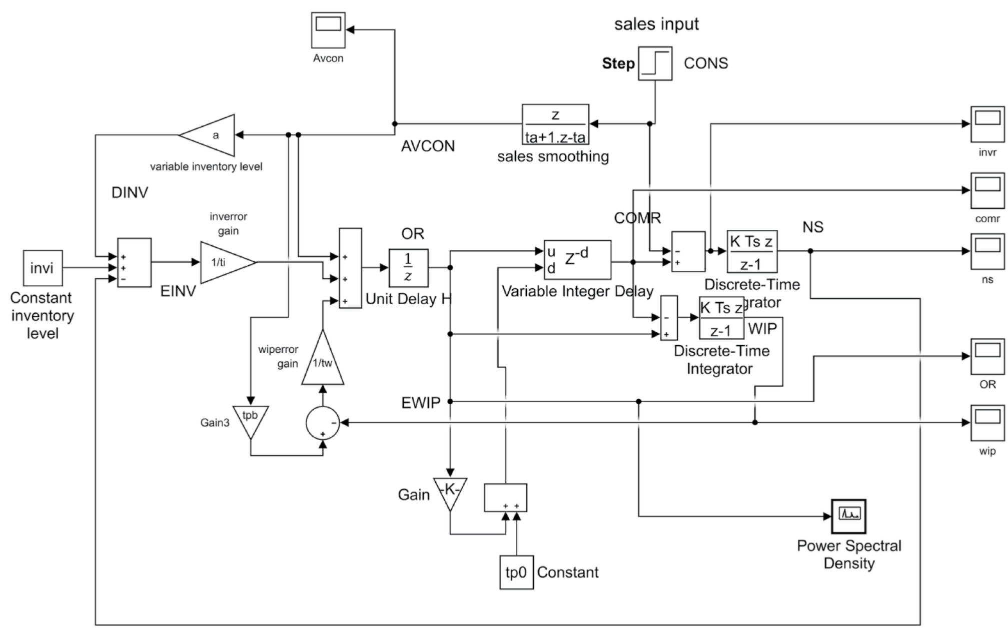
Tests for Chaos

To detect chaos, Liapunov exponents derived from the eigenvalues of the system are used. These are a measure of the average rate of convergence or divergence of adjacent orbits in phase space (Wolf et al., 1985). "Any system containing at least one positive Liapunov exponent is defined to be chaotic, with the magnitude of the positive exponent reflecting the timescale on which system dynamics become unpredictable." Wolf et al. (1985) provided one of the most used techniques for calculating the exponents from real measured data.

However, in this work the simpler technique to extract the largest of the Liapunov exponents λ_1 developed by Rosenstein et al. (1993) is used as follows.

It is assumed that the *jth* pair of nearest neighboring paths in phase space diverge approximately at a rate given by the largest positive Liapunov exponent:

Figure 3. Simulink model for APVIOBPCS with variable delay



$$d_j\left(i\right) \approx e^{\lambda_1\left(i\Delta t\right)} \tag{14}$$

Solution of a first order set of differential equations path (Edwards and Penney, 2004) is $Ce^{\lambda t}$. The divergence, in phase space at each time interval of an initial separate pair of point will also be a similar function and:

$$ln(d_j\left(i\right)) \approx lnC_j + \lambda_1\left(i\Delta t\right) \tag{15}$$

We can define:

$$y\left(i\right) = \frac{1}{{}^{\prime\prime}t}\ln d_j\left(i\right) \tag{16}$$

Equation 15 "represents a set of approximately parallel lines (*for j* = 1…*M*), each of whose slope is roughly proportional to λ_1". Equation 16 shows the largest Liapunov exponent is computed from the average *of j* path pairs using a least-squares fit.

If the graph of log (divergence) is a straight line of positive slope then the system is defined as chaotic. Hilborn (1994) gives system classifications for various values of Liapunov exponent illustrating that for an exponent of value 0 the motion is quasi-periodic and a for an exponent of positive value chaotic behavior occurs. Several routes that lead to chaos may be present including period doubling via quasi-periodicity.

NUMERICAL EXPERIMENTS AND RESULTS

In this section the methodology used is explained for the two sets of experiments that are described here. The first set deals with the responses of a discreet model of a single tier inventory under conditions of no order rate limits and those where the limits are set. The second set of experiments deal with the problem of process delays that depend on order rate, in this case with no capacity limits. Tests to validate the results are shown to give confidence in the outcomes.

Unlimited Orate Discrete Model

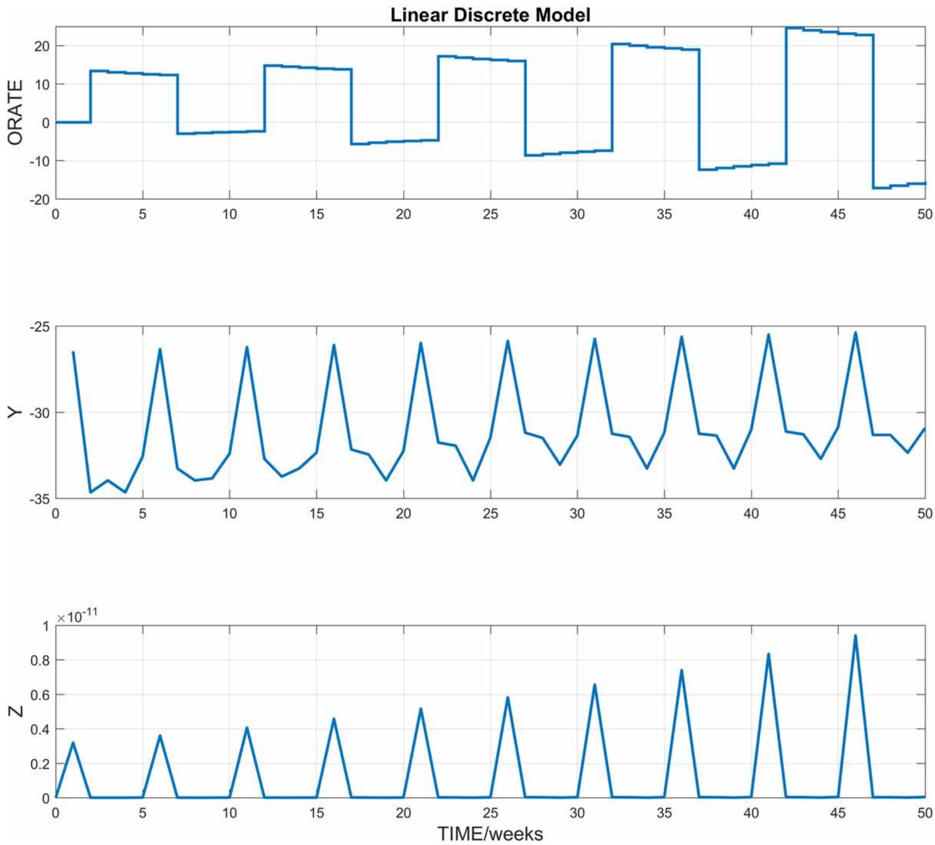
First the case of the discrete simulation model without capacity limits operating in an unstable linear region is examined. This instability is indicated by Disney & Towill (2002) and White & Censlive (2006) for the parameter values *ti* = 0.47 and *tw* = 1. These values produced an unstable growing nearly square wave oscillation shown in Figure 4. The plot of Z, the divergence shows periods of zero value followed by triangular peaks of increasing magnitude. The plots of Y, log (divergence) show a clearly nonlinear oscillation of steadily increasing mean value.

The divergence between paths Z and the $\log_e$ (divergence), Y, both increase with time showing that the paths are separating in phase space, a critical indicator of chaotic behavior. The forms of the plots of the power spectral density and autocorrelation functions, shown in Figures 5 and 6, give added evidence for chaos. The power spectrum has a level of 50-70 dB with superimposed broad peaks. The autocorrelation diagram (Figure 6) shows an oscillation dying away rapidly either side of zero.

Figure 6 gives an autocorrelation diagram showing a triangular shape similar to that of a square narrow pulse.

The pole-zero map for this data is shown in Figure 7. This is an Argand diagram obtained from the discrete transfer function with the poles as values that make the transfer function tend to infinity

Figure 4. Time response and Liapunov Index for linear discrete model, it = 0.47, tw = 1, tp = 4, ta = 8, n = 10



and the zeroes values that make the transfer function zero. For a stable system the eigenvalues need to be inside the unit circle. The poles and zeros are for the forward path transfer function, whereas the eigenvalues are for the closed loop system.

The average Liapunov exponent for this data is 0.4, which corresponds to a z value of 1.49, between the RHS pole and zero.

Another case here shows the average Liapunov exponent is 0.1853 corresponding to a z value of 1.204. Examining the pole zero map for this data shown in Figure 8 the Liapunov value is between the pole and zero on the RHS.

Limited Discrete Model

For the model with orate limits we can have two cases to consider:

1. The response plot tends to cross the zero minimum orate boundary (n = 5);
2. Where the response tends to cross the orate capacity upper limit (n = 10).

Laugesan & Mosekilde (2006) have found evidence of chaos due to the response crossing the zero orate boundary. It is possible that this condition could also be found at the upper limit to production in a real system. A new investigation was conducted with the two cases using the same base values as for the unstable discrete model, using the Liapunov exponent criterion for chaos.

Figure 5. Power spectrum of orate response for unlimited discrete model

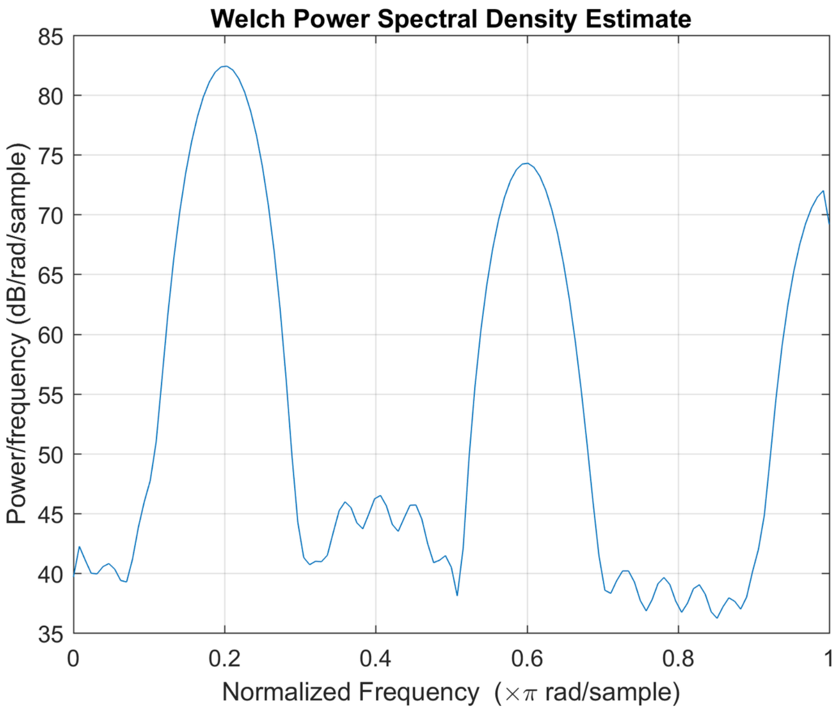


Figure 6. Autocorrelation diagram for unlimited discrete model

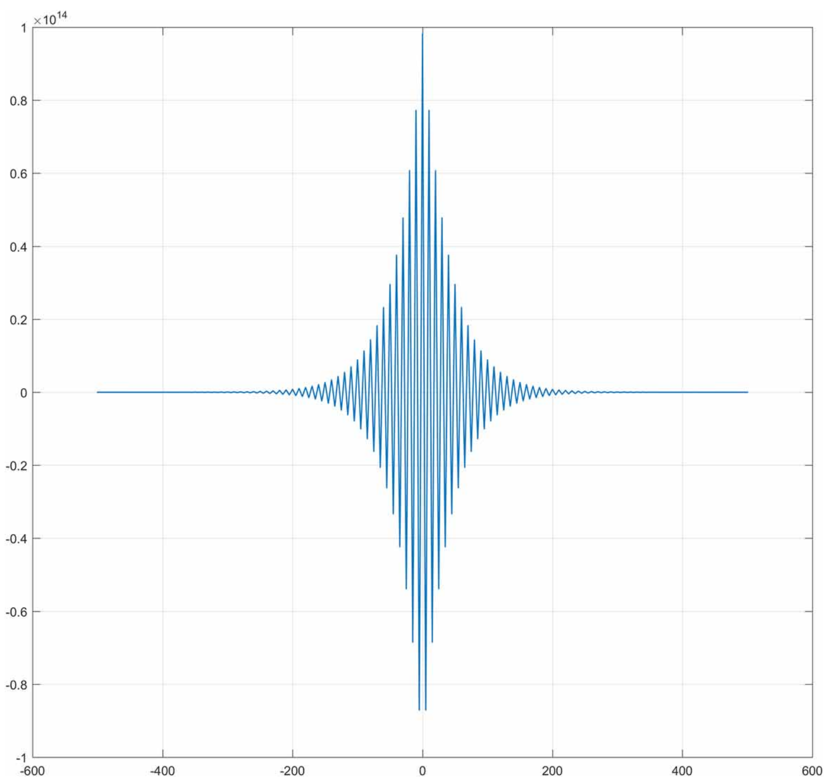
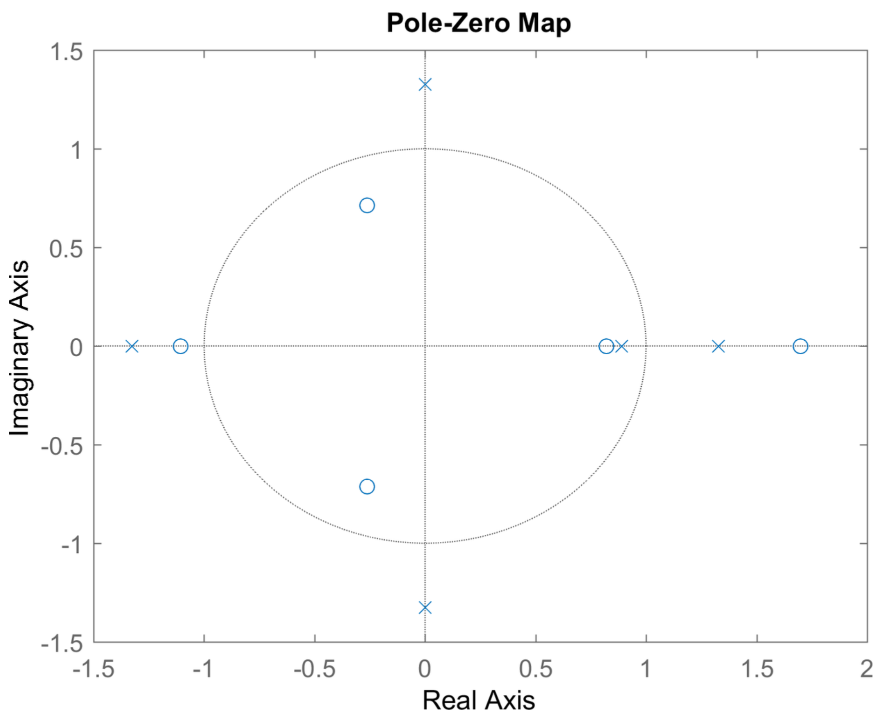


Figure 7. Pole zero map for unlimited discrete model



1. The response impacting the lower ORATE boundary with n = 5

A stable discrete limited model shows a convergence (Figure 9). A plot of divergence Z has a small but negative slope of the order of $-7*10^{-9}$ but the plot shows considerable oscillations.

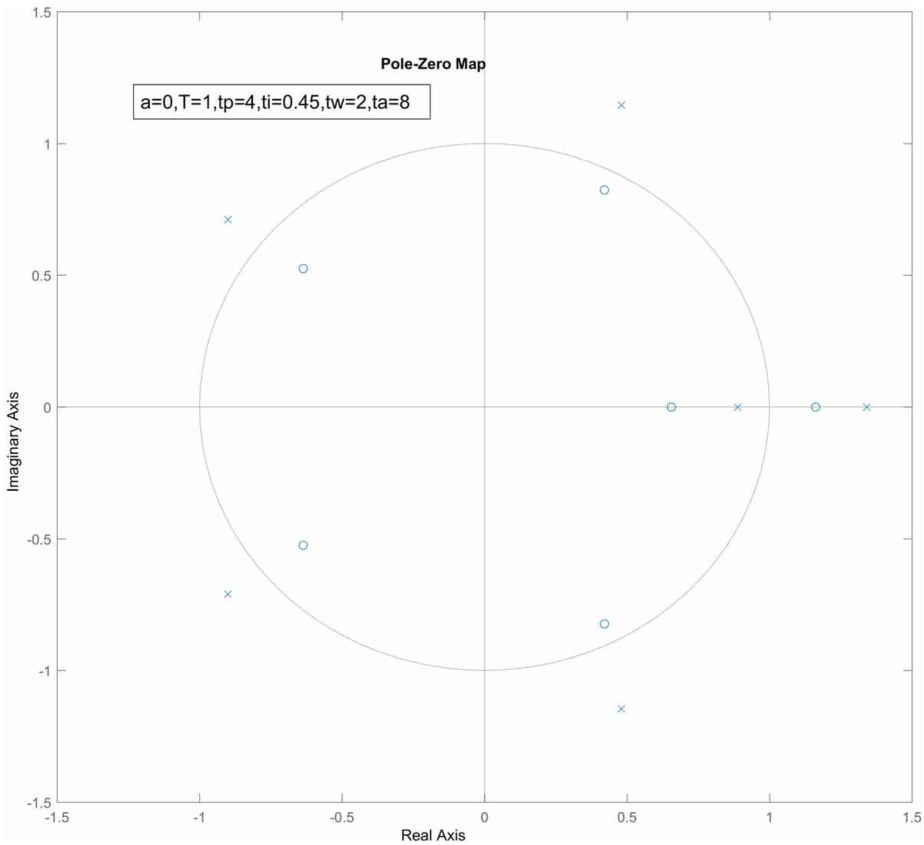
In Figure 10, the ORATE curve is a limited square wave with minor oscillations. It shows that although there are gaps in the log (divergence) plot corresponding to the zero values of the path difference, when the ORATE stays at the upper limit, the trend is to an increasing value of path divergence. The power spectrum (Figure 11) shows a curve with several peaks, not typical random noise as it drops below 0 db. The autocorrelation function (Figure 12) is also not typical of just random noise but is more typical of a set of sine waves (which is also the autocorrelation plot expected from a set of square waves).

2. The second example is where n = 10 and the response plot hits the upper limit of maximum capacity (Figure 13), with the corresponding power spectrum in Figure 14 and the autocorrelation diagram in Figure 15. The log (divergence) Y, has significant gaps corresponding to the value of ORATE staying at the upper boundary. The overall change as time progresses is still an increasing average value, suggesting chaos.

In comparison we can see that for a continuous limited model with the same base conditions does not show the value of X or Y slowly increasing, so no signs of chaos (Figure 16).

Using data from Wang et al. (2005) for a longer simulation period (Figure 17) shows that the log curve does not increase continuously with time. For simulation times less than 60 weeks a positive Liapunov exponent indicates chaotic behavior but at around 60 weeks the slope of the Y curve tends to zero, indicating a change in system dynamics. This is a significant result, discussed below.

Figure 8. Pole zero map for ti = 0.45



Variable Delay Time

In this section we present the results from simulations run with a variable time delay with examples of checking the method. These are easier to see than the same tests carried out on the earlier model as they are a continuous record.

Applying the test that increasing log (divergence curves) with positive slope indicate chaos, to Figure 18, shows that for a stable continuous model no chaos is present as the divergence curve is of negative slope.

Procedural Tests

The procedure was repeated several times with the same data to test to see if this phenomenon is real or an artefact. It is shown in Figure 19 that the positive gradient indicating a divergence in phase space does not depend on the distance apart of the initial points or the number of averages curves which are included in the averaging process. The curves show a continuous curve with oscillations about a straight line. The dependency on the number of averaging sets of data is shown in Figure 20, here the slope is essentially the same with only a small change to the initial condition. The effect of integration routine using three types of discrete approximation is indicated in Figure 21, again the slope is essentially unchanged. When the sampling time is varied in Figure 22 shows the same general results of a positive slope were obtained. However, the amplitude of the oscillation increases with the sampling time.

Figure 9. Convergence of data for a stable limited model

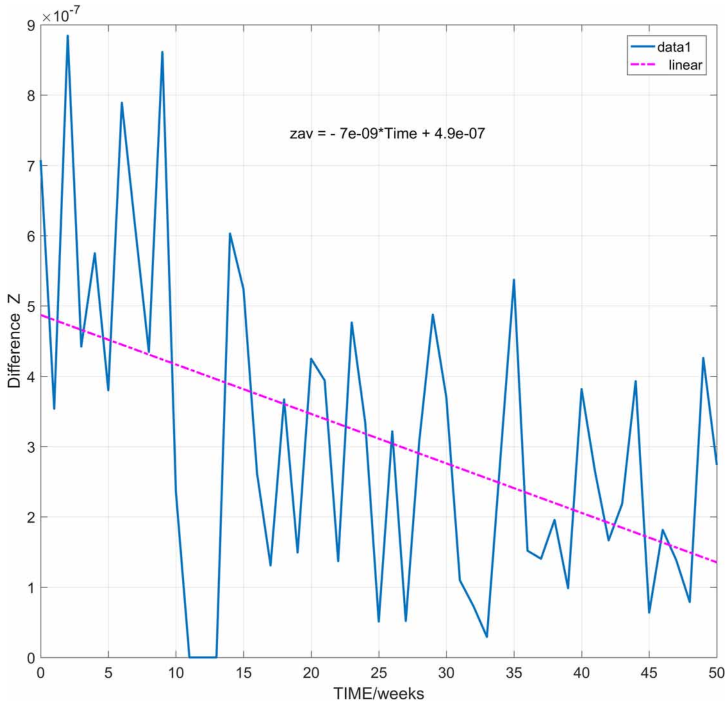


Figure 10. Response for discrete model with n=

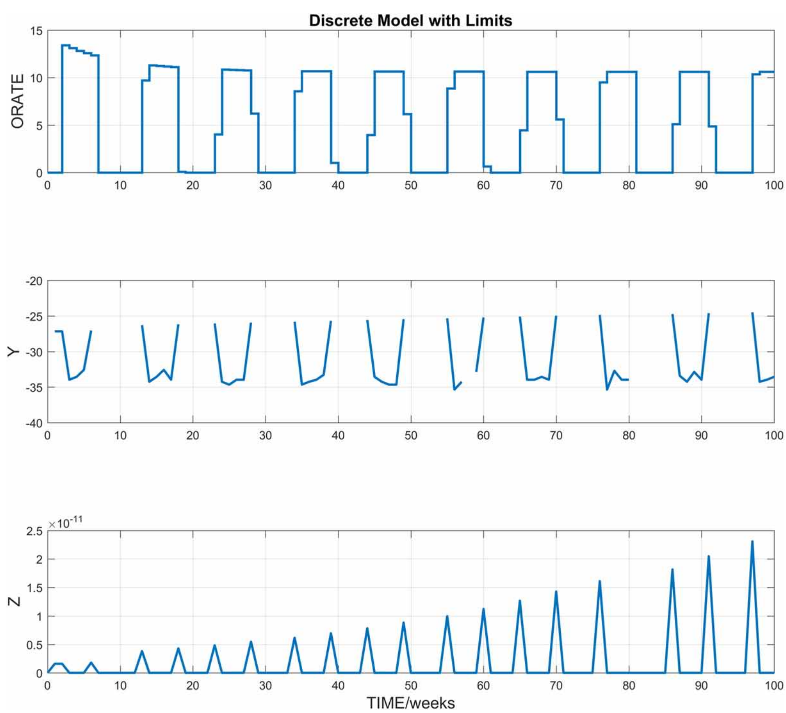


Figure 11. Power spectrum n=5

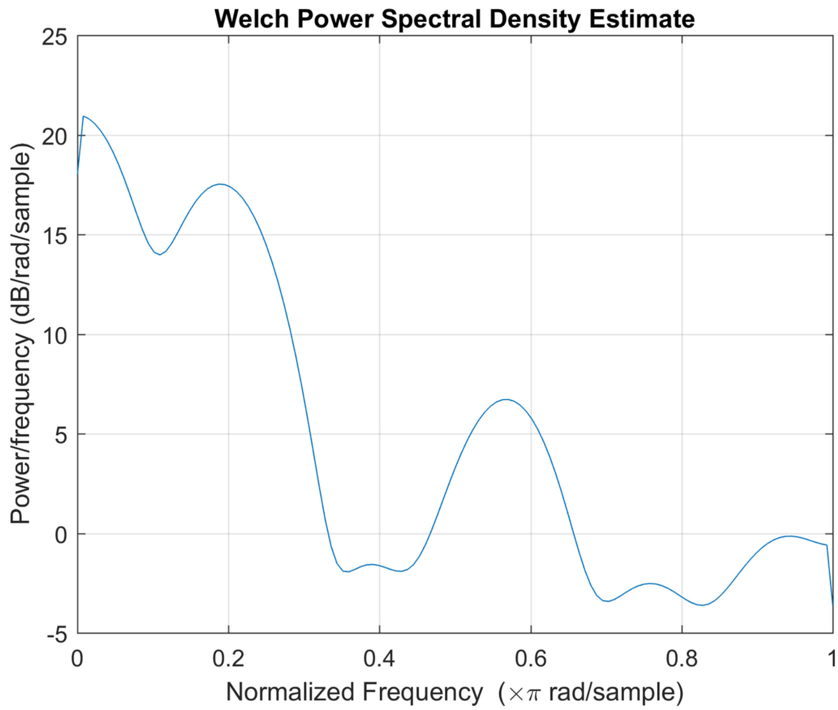


Figure 12. Autocorrelation for n=5

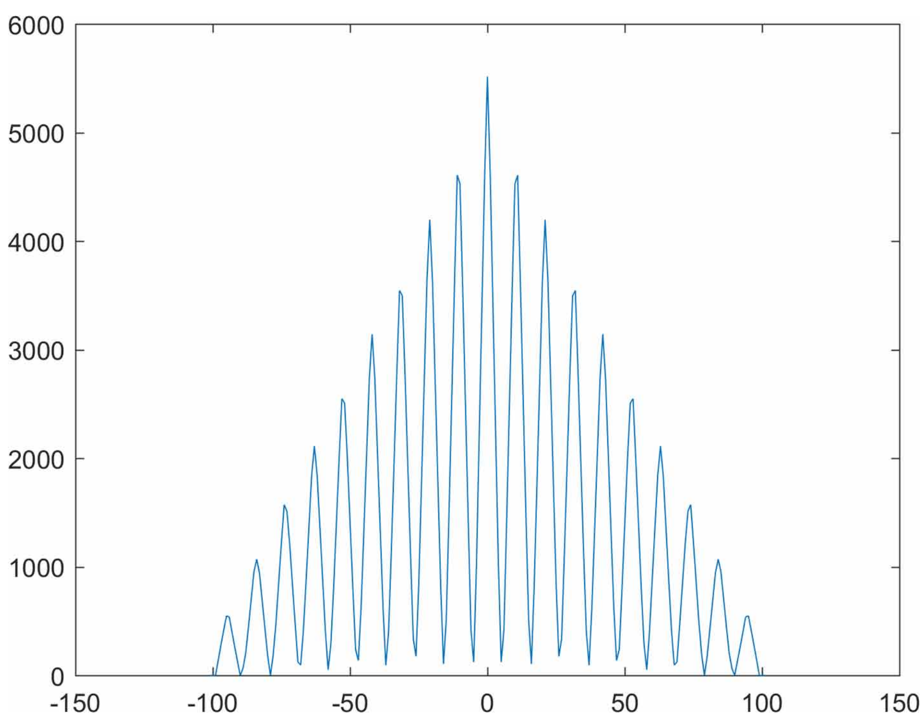
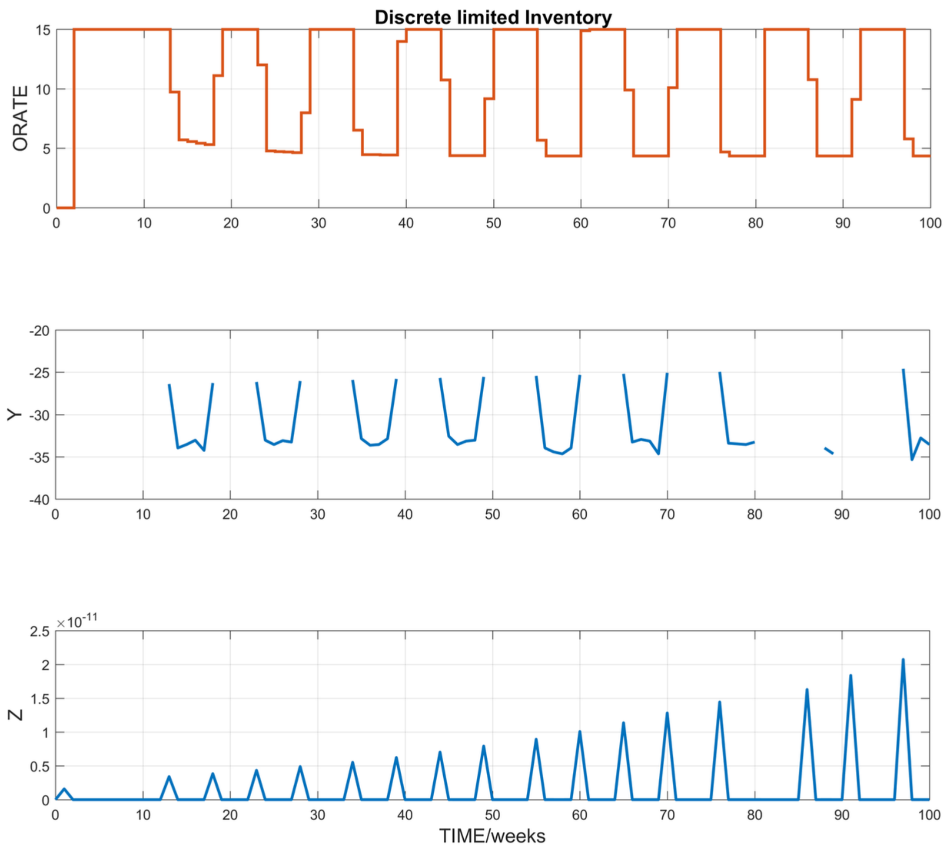


Figure 13. Response of limited discrete model for n = 10



All the curves in this section show a positive average curve slope with a ripple superimposed on the linear curve. The observed differences in slope are very small for all the conditions described in this section except for the plots showing effect of variation of sample time. In Figure 22 the oscillations are largest with the largest value of sampling time. While in Figure 23 as the sample time is reduced to make the system closer to a continuous system the value of the Liapunov exponent is near zero.

Results for Varying Time Delay

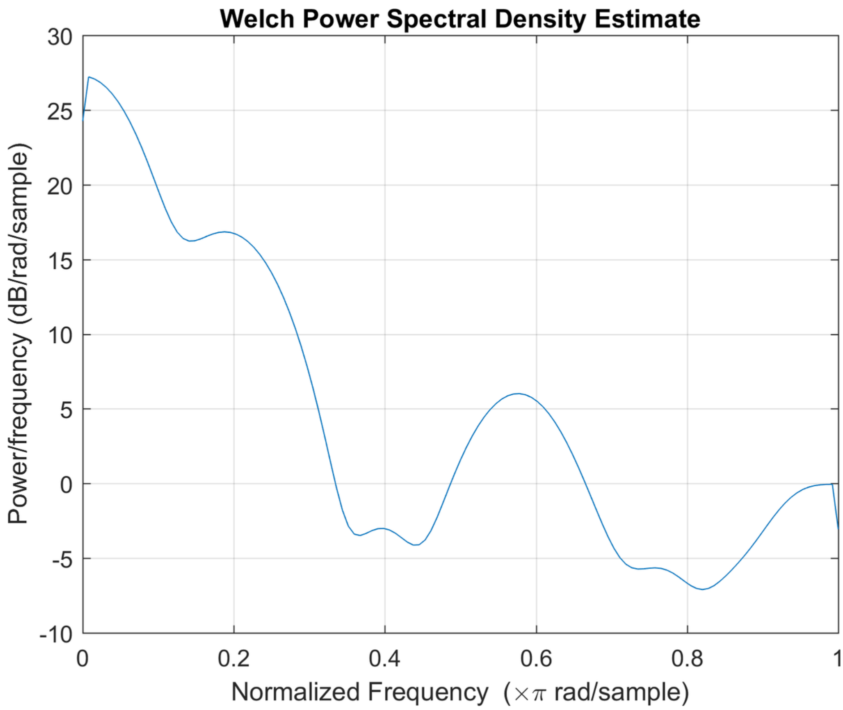
The responses of the varying time delay system produced a wide range of similar results. The plots shown in Figure 24 are typical.

The base result for $T = 1$, $tp0 = 4$, $tp1 = 0$ is shown as a solid line. The slope for $tp0 = 3$ is greater while introducing a component proportional to the order rate the plot still has the positive slope but at a reduced level.The best fit straight lines are given by *yp*.

Effect of Inventory Model Structure

The results of Y, shown previously (Figures 5 to 24) are for the APIOBPCS models with the parameter a is set to zero. But if $a \neq 0$ then the model becomes that for an APVIOBPCS system. The results shown in Figure 25 indicate that the responses are virtually identical to those of the APIOBPCS model. For comparison results from the much simpler structured model IOBPCS are included in Figure 26, producing a similar shaped curve to those of the APIOBPCS, but of greater slope.

Figure 14. Power spectrum for n = 10



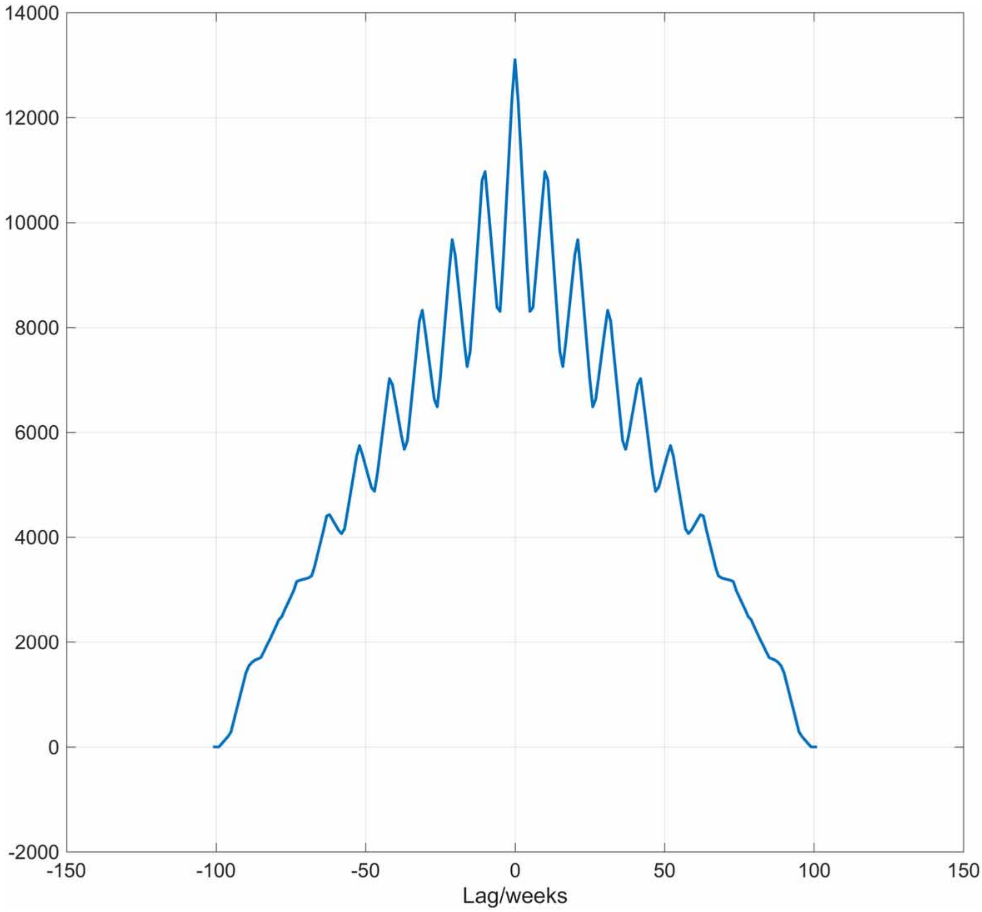
DISCUSSION

The first issue of note is that for a stable continuous or discrete system without limits (in ORATE or other factors) no evidence of chaos is observed in the response of these models for a sudden step demand in sales. For an unstable discrete system without limits chaos is indicated by an increasing average of the log(divergence) curve (Figure 4). The power spectrum has several peaks but also a broad noise base again tending to support the presence of chaos. There are no sharp spikes in frequency that would be expected of a periodic system. The nonlinearity in this system would likely be in the modelling of the sampling process itself.

For the limited inventory the evidence is as confused as the nonlimited case. With the impacts on the lower order rate boundary the average divergence curve again increases but the log curve has gaps where the response impacts the boundary (Figure 10). The case of a continuous model where the parameter values are chosen to be the same for the unstable discrete system shows no sign of increasing log (divergence) curve and so the system behaviour is judged not to be chaotic. The impact on the upper boundary (Figure 14) where there is a limit to capacity shows an increasing divergence but the log plot shows an average slope essentially zero, implying no chaos. However, the power spectrum and autocorrelation plots do not rule out chaos!

Using data from Wang et al. (2005), which are reported to be chaotic, the log diffusion plot shown in Figure 17 indicates an initial region of increasing log (divergence) values indicating a chaotic regime followed after 60 days by a period where the indication of chaos disappears. For the period from 60 to 200 days the system is apparently aperiodic. This form of result is also reported by Rosenstein et al (1993) for other systems. When the results are examined for the variable delay time the picture is clearer. Chaos is indicated for a discrete limited system (Figures 19-26). For comparison Figure 18 shows that for a stable continuous model the system response shows no chaos. Further detailed simulations (Figures 22 and 23) show that the oscillations in the log (divergence) curve are

Figure 15. Autocorrelation for n = 10



due to the discrete step size, virtually disappearing for a sample size of $T = 0.05$. This effect was also observed for the unlimited model. It is also clear that the slope of the log (divergence) curve depends on this sample time, with the log curve slope reducing as the sample time reduces, getting closer to the continuous case. It is also clear that the APVIOBPCS and APIOBPCS model topologies give essentially the same results and are indistinguishable from each other in overall effect. This is not true for the IOBPCS and APIOBPCS models. In this case the IOBPCS has a greater slope for the log curve leading to a faster divergent chaotic response, so that an IOBPCS structure is more likely to give a chaotic response.

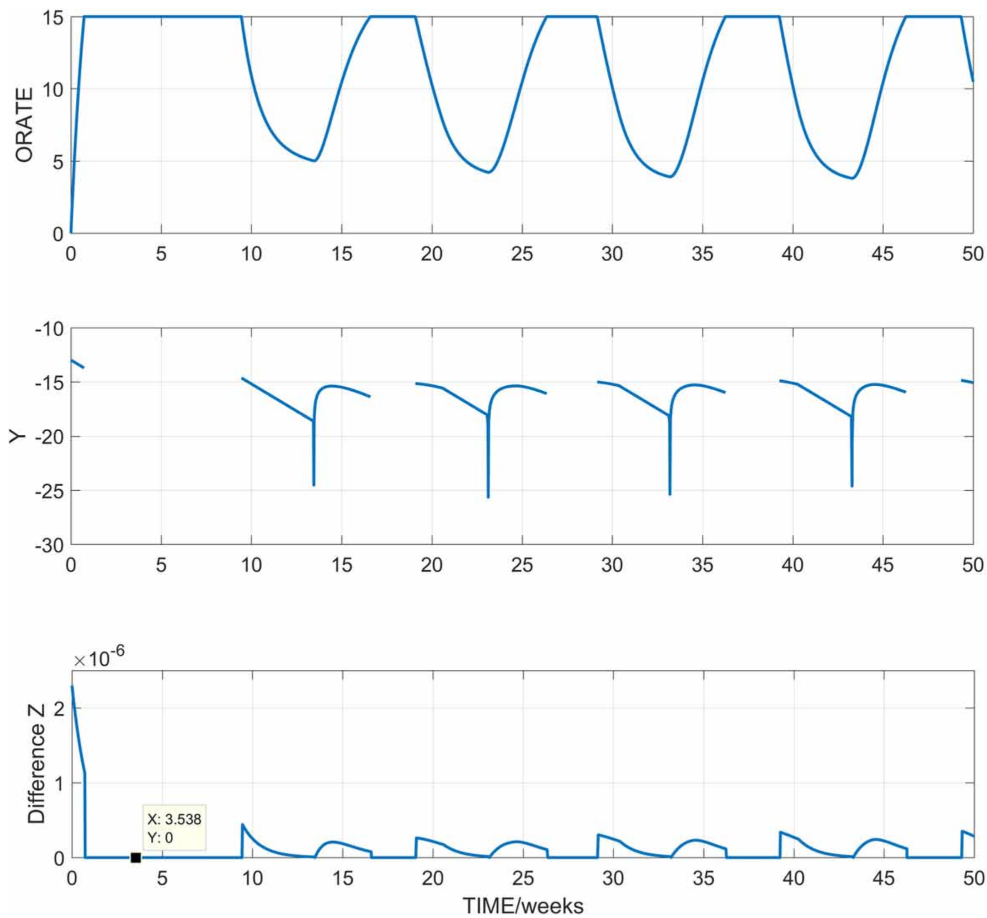
The effect of setting the time delay in proportion to ORATE value actually lessens the tendency to chaos as evidenced by the smaller values of the Liapunov exponents, while increasing the fixed delay time lessens the tendency to chaos in an unstable limited system.

This suggests that more investigation is needed to see whether a slow onset of chaos is indicated.

Generally, results shown here support that of other workers that chaos doesn't exist for values of positive *tw* above the line *tw* = *ti*. There is contradiction with observations of chaos at the limits. This needs further work to resolve the reasons for this behavior.

The observation that when the unlimited model is changed from a discrete version to a continuous model and if the sampling time in the discrete version is reduced indications of chaos disappear is highly significant. It may be that deterministic chaos does not exist in real supply chains and only

Figure 16. Continuous limited model



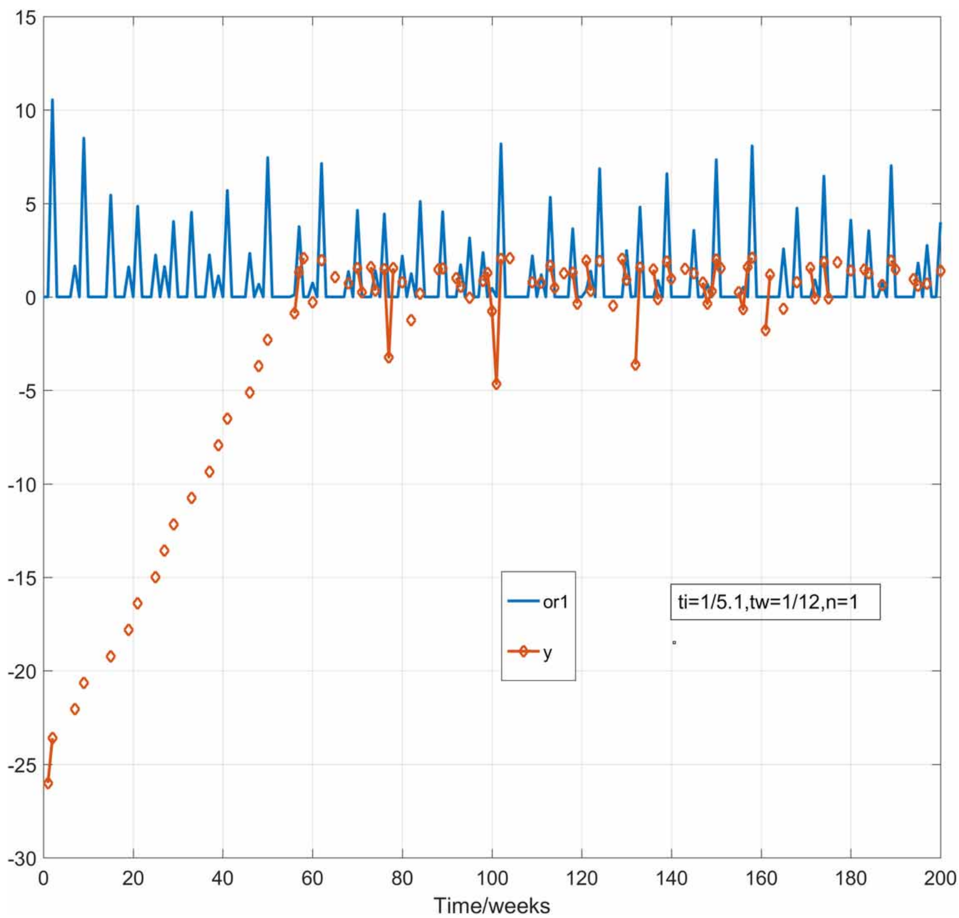
in the models or that in real supply chains it is caused solely by having orders processed at specific intervals. The implications for managers is that by processing orders as they come in the likelihood of chaos is substantially reduced.

Results of Figure 17 however pose a different problem. If the response is due to the startup transient oscillation of the system model then this leads to difficulties trying to respond to agile production as the system would be continually in this state. This means that having the values of *ti and tw* set to values that give a stable system is essential for agile response.

CONCLUSION

APVIOBPCS control theoretic models used by Disney and Towill in the z transform format are examined here. They are formulated to conform to industrial practice and are designed to give the best response to sales inputs with minimum stock out using proportional feedback of inventory error, based on actual observations of a real system. Note that for stable continuous models there is no chaos present and for unstable linear discrete systems chaos is apparently possible for systems with finite sample times greater than 0.1 units. The occurrence of chaos was found to be related to the sampling time of the discrete system, the magnitude of the Liapunov exponent reducing as the sampling time

Figure 17. Discrete limited model using data from Wang et al. (2005)



was reduced, indicating that chaos was less probable. For systems with capacity limits we could not confirm chaos at the no negative order boundary which are at odds with other researchers.

When a system of varying delay time was investigated the effect of increasing the dependence on orate reduced the effective Liapunov exponent. Increasing the delay time also reduced the Liapunov exponent. This leads to the hypothesis that modifying the manufacturing approach to try and minimize the production delay as might be the modus vivendi in an agile manufacturing operation could put the system in danger of slipping into chaotic behavior. The results obtained here also indicates that making the production delay depend on the volume of orders is likely to reduce chaotic behavior.

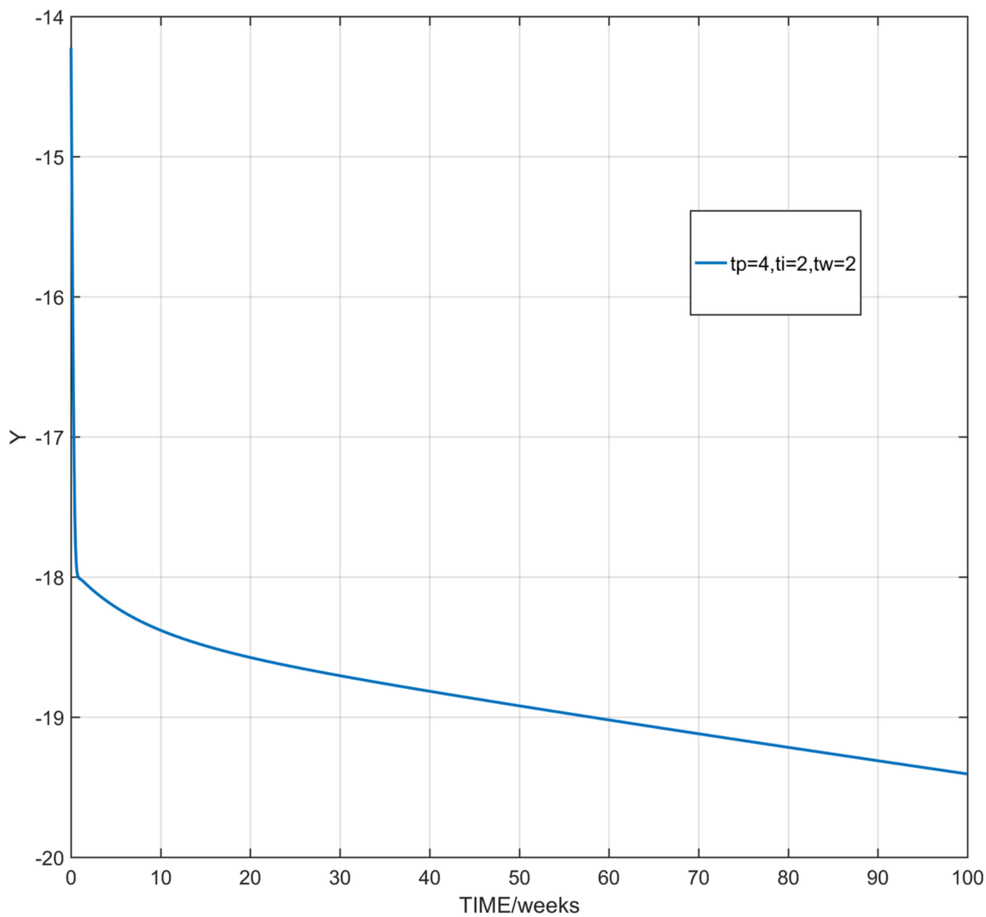
Model structure has some influence on the values of the Liapunov exponent with the largest effect arising in the simpler IOBPCS model.

The state space path divergence effects between the APVIOBPCS and APIOBPCS models are seen to be indistinguishable.

Application of Results to Organization and Management Actions for the Real World

The results suggest that the tendency to chaos in real world supply chains can be reduced by the following actions:

Figure 18. Stable continuous model with variable delay dependent on Orate



- Organize the real-world business system structure so that it resembles an APIOBPCS structure;
- Reduce the production and process delay time (as widely practiced) but not less than twice the order update time;
- Use an order update frequency as short as possible;
- Make realistic delay times dependent on ORATE volume.

Future Work

- Extend the range of parameter values to include studies of chaotic and hyperchaotic examples reported elsewhere;
- Determine the bounds of chaotic behavior for the case of “no return of orders”;
- Extend this investigation to include producer; warehouse and distributor levels as well as the retailer;
- Extend analysis to other models of delay dependency;
- Extend the analysis to considering non-proportional inventory and WIP demand;
- Compare, if possible, the data produced by models with real world data.

Figure 19. Variable time delay with different initial starting separations

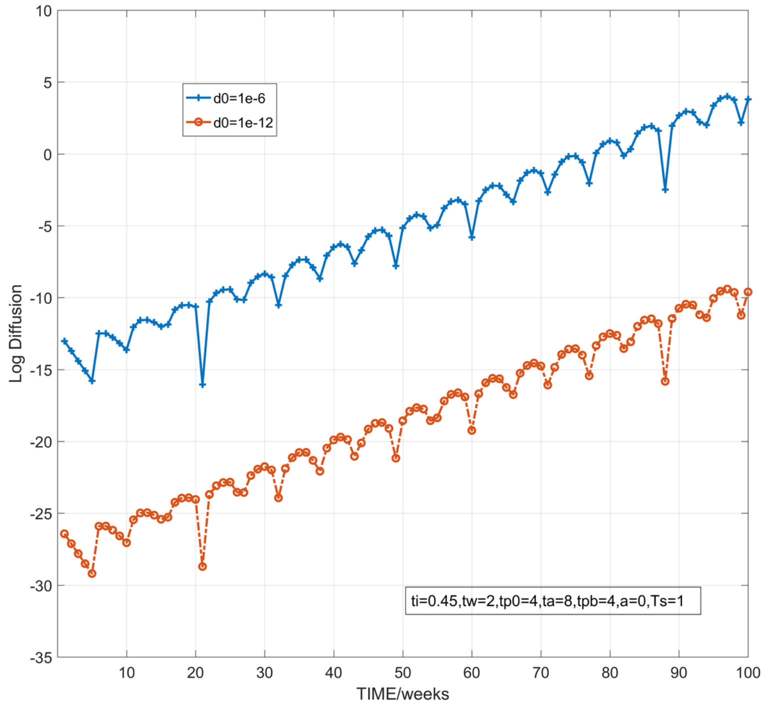


Figure 20. Variable time delay with numbers of averaging paths in state space

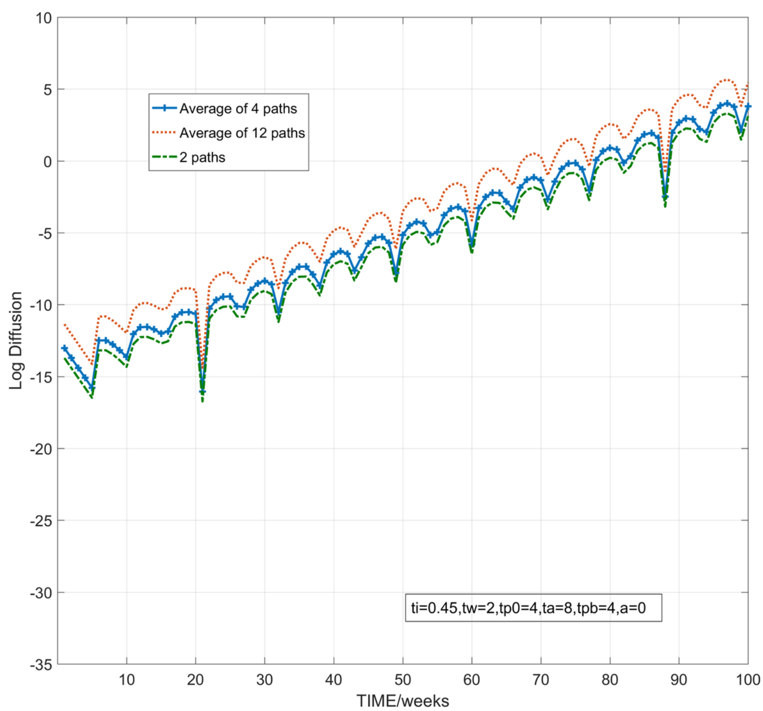


Figure 21. Variable time delay with different Simulink integration routines

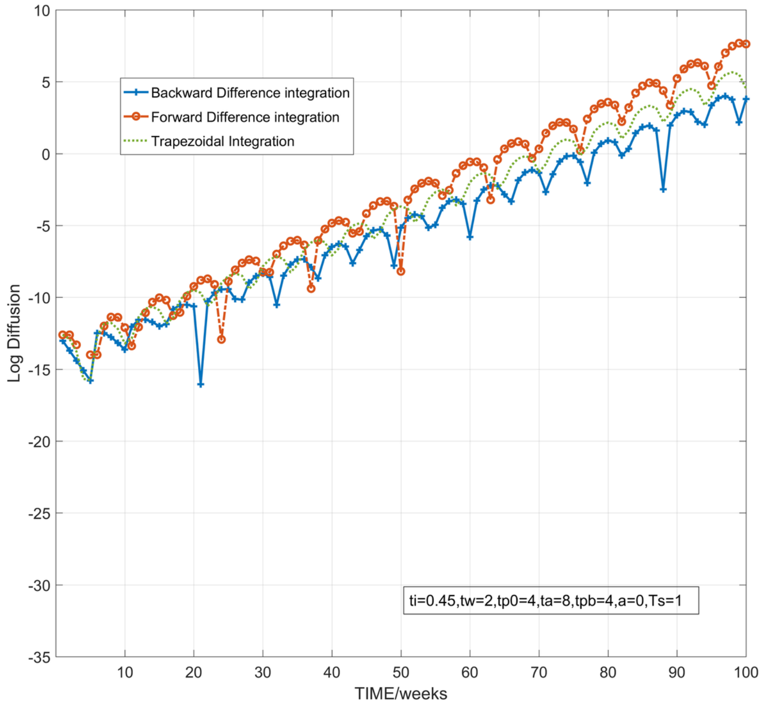


Figure 22. Effect of variable time delay with different sample times

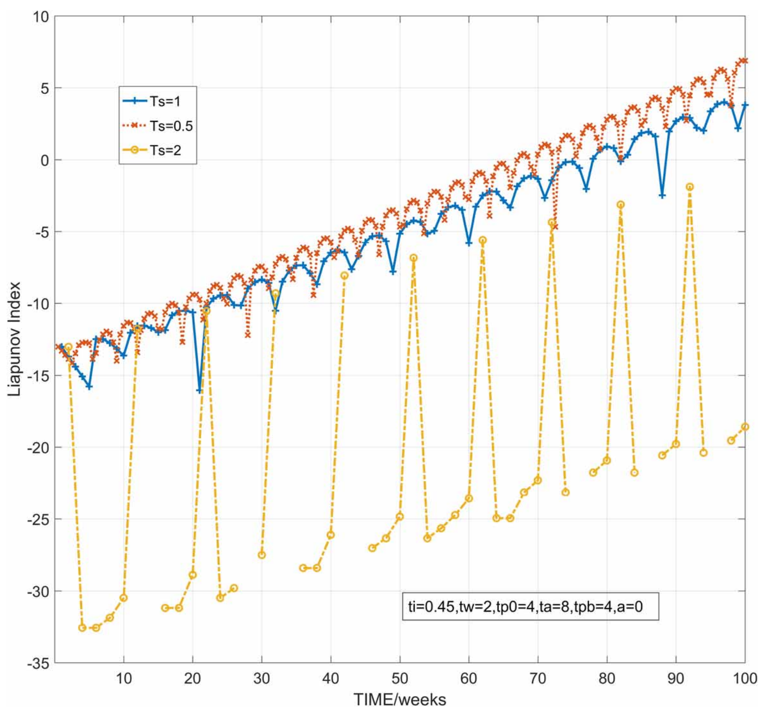


Figure 23. Effects of varying sampling time

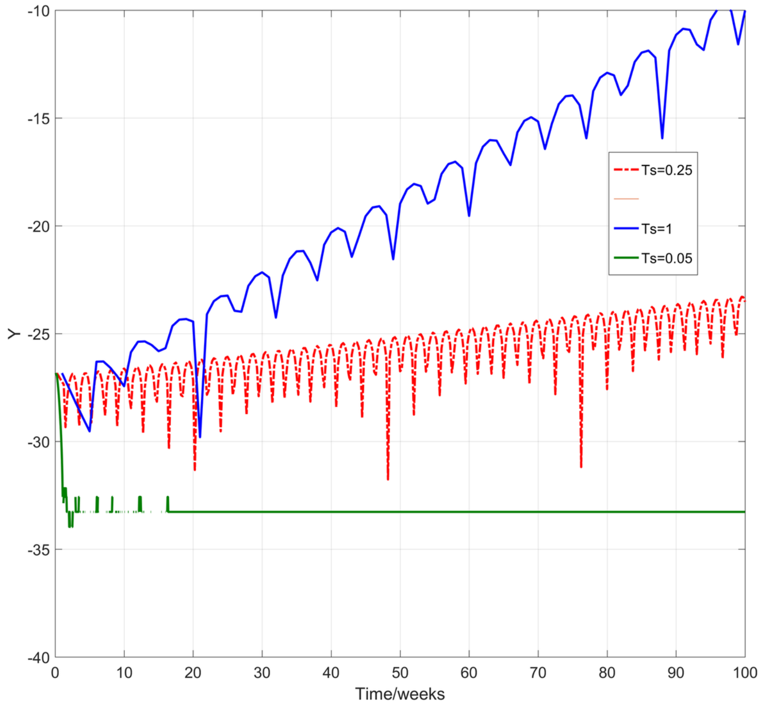


Figure 24. Effects of varying time delay

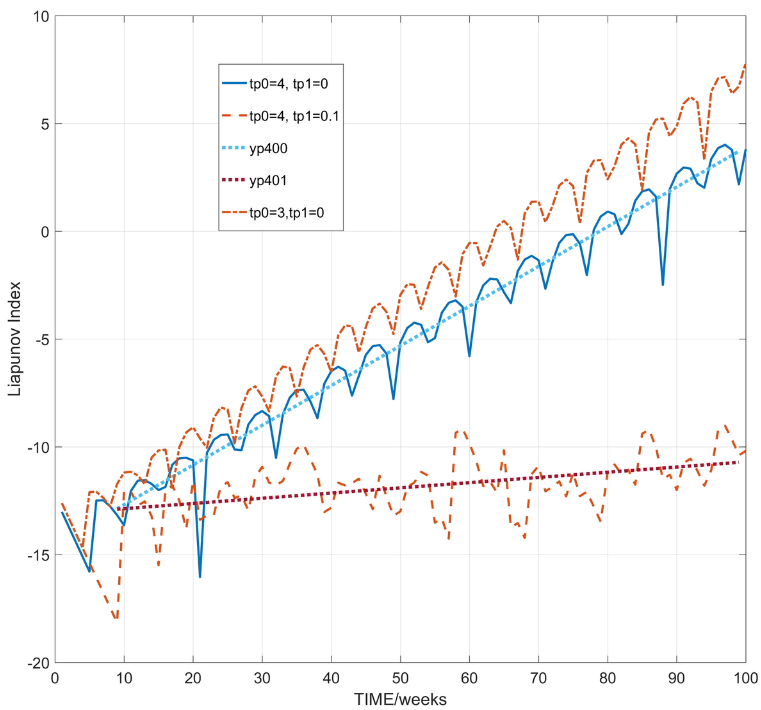


Figure 25. APIOBPCS and APVIOBPCS model results

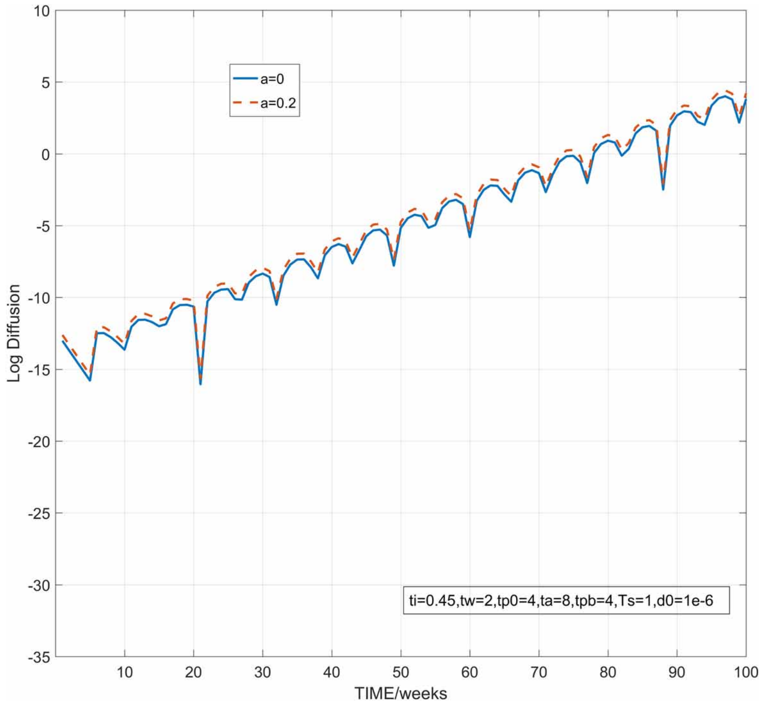
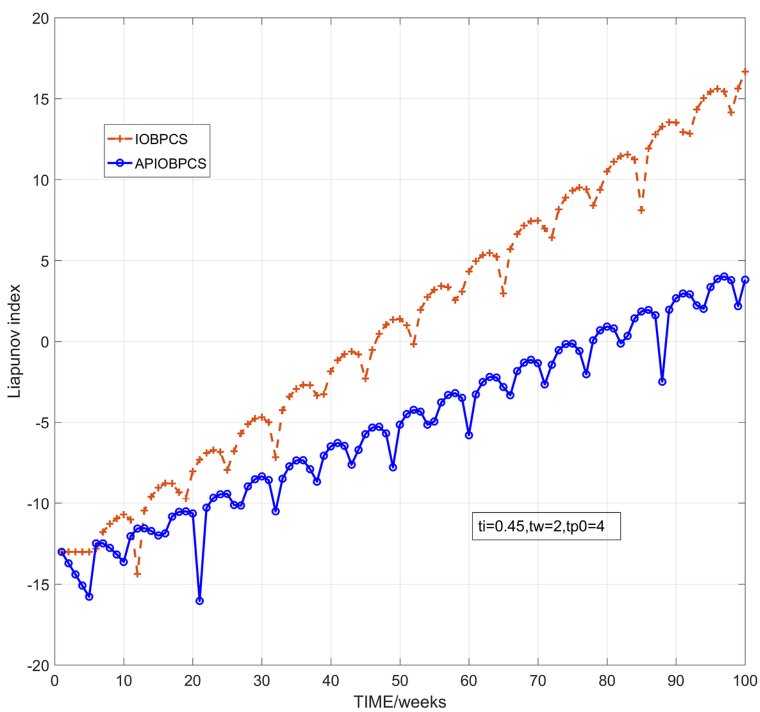


Figure 26. Effects of different inventory model structures



Symbols

α: Fraction of AVCON fed forward
APVIOBPCS: Automatic Pipeline Variable Inventory and Order Based Production Control System
AVCON: Average sales rate
CONS: consumption or market demand
EINV: Error in inventory level
DWIP: Desired work in progress
ORATE: Outstanding level of orders placed with the supplier
COMRATE: Rate of production
i: Number of discrete time steps Δt
IOBPCS: Inventory and Order Based Production Control System
n: Size of step input
TINV: Target inventory
T: Sampling time
t_a: Smoothing time constant (8 weeks)
t_i: Order constant time. (1 weeks)
t_p: Production delay time (4 weeks)
t_w: WIP delay time (2 weeks)
Y: average Log_e (divergence)
Z: Path Divergence

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